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THE RISE AND FALL OF TYPED SENTENCES

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Abstract. We characterize the 3-stratifiable theorems of NF as a 3-stratifiable extension of NF_3; and show that NF is equiconsistent with TT plus raising type axioms for sentences asserting the existence of some predicate over an atomic Boolean algebra.

§1. Setting. TT is the theory of types associated with NF, Quine's *New Foundations*. While NF is a first order theory, TT has its variables typed with the natural numbers and its logic modified accordingly. We recommend Thomas Forster's book [5] as the standard reference on these topics; and Randall Holmes's NF page `math.idbsu.edu/~holmes/holmes/nf.html` as the best Internet site. We warn the readers, if any, that the use of types in written language can become a bit messy; after all that's why typical ambiguity and NF were invented! Fortunately people used to TT often grasp things more easily by drawing pictures with horizontal lines and arrows than in writing.

$\mathcal{L}_{TT}$ is the language of TT and $\mathcal{L}_{TT_n}$ is the language of TT_n, $\mathcal{L}_{TT}$ restricted to the first n types: $\{0, 1, \dots, n-1\}$. In general, if α is a set of formulas in $\mathcal{L}_{TT}$, α_n is the intersection of α and $\mathcal{L}_{TT_n}$.

We write $\vdash$ for the derivability relation in $\mathcal{L}_{TT}$ or $\mathcal{L}_{NF}$ and $\vdash_n$ for the derivability relation in $\mathcal{L}_{TT_n}$.

If α is a variable, a formula or a set of formulas, α^+ is obtained from α by raising all the types by 1. α^k is $\alpha^{+++\cdots}$, where the $+$ operation is iterated k times.

A formula is n-typed if all its types are among a sequence of n consecutive types. Thus every formula of $\alpha_n, \alpha_n^+, \alpha_n^{++}, \dots$ is n-typed.

When, after possibly renaming bound variables to avoid unintended identifications, we erase the types in a set of formulas or in a formula of $\mathcal{L}_{TT}$, we obtain a set of stratifiable formulas or a stratifiable formula of $\mathcal{L}_{NF}$. The result of erasing types in such a way from α is denoted by $\overline{\alpha}$.

§2. Boolean algebras and models of TT_3. It is quite natural to associate with an atomic Boolean algebra, $\langle B, \leq \rangle$, the 2-typed structure $\langle A, \leq, B \rangle$ by taking the collection A of atoms as type 0. It happens that $\langle A, \leq, B \rangle$ is a model of TT_2. Conversely to any *model* of TT_2, $\langle M_0, \in_0, M_1 \rangle$, corresponds the atomic Boolean algebra $\langle M_1, \subseteq \rangle$ (see [7]).

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If $\langle B, U, \leq\rangle$ is an atomic Boolean algebra with a non-empty predicate U, one can associate with it the 3-typed structure $\langle U, \geq, A, \leq, B\rangle$, which is *not* in general a model of TT_3.

Conversely, we observe that any model of TT_3, $\langle M_0, \in_0, M_1, \in_1, M_2\rangle$, may be viewed as a model $\langle M_1, \in_1, M_2\rangle$ of TT_2 (an "atomic Boolean algebra") with an extra level M_0. The elements of this M_0 can be coded in M_2 as "Leibniz substances": the Leibniz substance of something being the set of all sets to which this something belongs.[1] It results that $\langle M_0, \in_0, M_1, \in_1, M_2\rangle$ may be viewed as an atomic Boolean algebra with a non-empty predicate $\langle M_2, U, \subseteq\rangle$, where U is the collection of the codes $\{x \mid a \in_0 x\}$ of the elements a of M_0.

Each sentence in the typed theory is therefore naturally translatable into a sentence in the theory of atomic algebra with a predicate and conversely.

Substituting typed structures for Boolean algebras, this justifies the following:

DEFINITION 1. *Let ϕ be a formula of $\mathcal{L}_{TT_4}$ and u a variable of type 3. Then $u^3 \models \phi$ is taken to be the formula constructed as follows: we replace each quantifier Qx^0 by the restricted quantifier $(Qx^2 \in u^3)$, the subformulas $x^0 \in y^1$ and $x^0 = y^0$ by $y^1 \in x^2$ and $x^2 = y^2$, and leave the others $(x^i \in y^{i+1}, x^i = y^i)$ as they stand. Of course, to prevent clash of variables, we have to suppose that variables like x^0 and x^2 must not both occur in ϕ. This can always be effected in case ϕ is a sentence by changing bound variables.*

The formula $u^3 \models \phi$ expresses that ϕ is true when we replace type 0 by u, and interpret the relation between u and type 1 as the converse of the former $\in$-relation. $u^3 \models \phi$ is 3-typed with 1, 2 and 3.

$u^{3+k} \models \phi$ stands for $(u^3 \models \phi)^k$, and since type 0 is not present in $u^3 \models \phi$, there is a formula, denoted hereafter by $u^2 \models \phi$, such that $(u^2 \models \phi)^+$ is $(u^3 \models \phi)$.

LEMMA 1. *Let*

$$\mathcal{M} = \langle M_0, \in_0, M_1, \in_1, M_2\rangle$$

be a model of TT_3 and U be non-empty and belonging to M_2. We extend $\mathcal{M}$ downwards by adding U as a new level and by defining the relation between U (the new type 0) and M_0 (the new type 1) as ${}_0\!\ni$, the converse of $\in_0$. We thus obtain a 4-typed structure:

$$\mathcal{M}_{\langle U,2\rangle} = \langle U, {}_0\!\ni, M_0, \in_0, M_1, \in_1, M_2\rangle.$$

(**1**) *If ϕ is a sentence in $\mathcal{L}_{TT_4}$, then, assigning U to the variable u^2, $\mathcal{M}_{\langle U,2\rangle} \models \phi$ iff $\mathcal{M} \models (u^2 \models \phi)$.*

(**2**) *If A is a closed axiom of comprehension in $\mathcal{L}_{TT_4}$ of the kind $\exists y^3 \forall x^2 (x^2 \in y^3 \leftrightarrow \phi)$ (y of the fourth type), then $\mathcal{M}_{\langle U,2\rangle} \models A$.*

PROOF. (**1**) is almost trivial as $\mathcal{M}_{\langle U,2\rangle} \models \phi$ is simply another way of saying that $\mathcal{M}_{\langle U,2\rangle} \models (u^3 \models \phi)$, which is equivalent to $\mathcal{M} \models (u^2 \models \phi)$.

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[1] Leibniz repeatedly suggests that a substance can be associated with the collection of predicates attributed to it. Thus he writes in *Discourse of Metaphysics* ¶*8*: " ... la nature d'une substance individuelle ou d'un estre complet est d'avoir une notion si accomplie qu'elle soit suffisante à comprendre et à en faire deduire tous les predicats du sujet à qui cette notion est attribuée.": " ... the nature of an individual substance or of a complete being is to have a notion so complete that it is sufficient to comprehend and to allow the deduction of all the predicates of the subject to which that notion is attributed." Forster [5] comments on the role of this important notion and refers to Boffa, Quine and Whitehead.

(2) $u^3 \models \exists y^3 \forall x^2(x^2 \in y^3 \leftrightarrow \phi)$ is $\exists y^3 \forall x^2(x^2 \in y^3 \leftrightarrow u^3 \models \phi)$. Hence $\forall u^3 u^3 \models A$ is provable in $TT_3^{\dagger}$. Therefore $u^2 \models A$ is true in $\mathcal{M}$ when U is assigned to u^2, and A is true in $\mathcal{M}_{\langle U, 2\rangle}$ by (1). ⊣

LEMMA 2. *$TT_4 \vdash_4 (\phi \to \exists u^3 u^3 \models \phi)$, for any sentence ϕ in $\mathcal{L}_{TT_4}$. $NF \vdash (\overline{\phi \to \exists u^3 u^3 \models \phi})$, for any sentence in $\mathcal{L}_{TT_4}$.*

PROOF. Let $\lambda(x^0)$ be the term $\{v^1 \mid x^0 \in v^1\}$, the Leibniz substance of x^0; and let U be $\{\lambda(x^0) \mid x^0 = x^0\}$.

Then, for any formula ϕ in $\mathcal{L}_{TT_4}$, it can be shown, by induction on the length of ϕ, that $TT_4 \vdash (\phi \leftrightarrow (U \models \phi)[\vec{x}^2 := \overrightarrow{\lambda(x^0)}])$, where the λ operation is applied to each variable in the list $\vec{x}^0$ of variables of type 0 that occur free in ϕ. This is essentially because $TT_4 \vdash x^0 \in y^1 \leftrightarrow y^1 \in \lambda(x^0)$ and $TT_4 \vdash x^0 = y^0 \leftrightarrow \lambda(x^0) = \lambda(y^0)$.

The proof for NF is similar. ⊣

§3. Ambiguity reduced.

Amb is the collection of ambiguity axioms: the sentences $\phi \leftrightarrow \phi^{\dagger}$.

It is known that **Amb**$_3$, i.e., ambiguity for 2-typed sentences, is true in every model of TT which is externally infinite (see [2] and [7]). Moreover it is a consequence of Grishin's reduction of TT to TT_4 ([6] is one of the original Russian references; see also [1] and [3]) and Specker's connection between NF and $TT +$ **Amb** (see [8]) that $TT +$ **Amb**$_4$ is equiconsistent with $TT +$ **Amb** and with NF.

Call a sentence $\exists x \phi$ a Σ-sentence when it is 3-typed and x is its unique variable of highest type. Whenever ϕ is a sentence in $\mathcal{L}_{TT_3}$, $\exists u^2 u^2 \models \phi$, $\exists u^3 u^3 \models \phi$, $\exists u^4 u^4 \models \phi$, ... are all Σ-sentences.

Now we introduce another kind of ambiguity axioms that look just a little stronger than **Amb**$_3$ but much weaker than **Amb**$_4$.

ΣAmb is the set of sentences of the form $\exists x \phi \to \exists x^{\dagger} \phi^{\dagger}$, for Σ-sentences $\exists x \phi$.

The schema **ΣAmb** expresses in particular the fact that if there is a predicate U over an atomic algebra constituted by 2 consecutive types, k, $k+1$, verifying a property expressible by a formula of the language of Boolean algebras with a predicate, then there is a predicate V over the algebra constituted by the two next types, $k+1$, $k+2$, satisfying the same property.[2]

ΛAmb is the set of sentences of the form $\phi \to \exists u^2 u^2 \models \phi$, ϕ in $\mathcal{L}_{TT_3}$. These sentences of $\mathcal{L}_{TT_3}$ mean that every model verifying ϕ can be extended downwards to a structure satisfying ϕ again by adding a level below level 0.

THEOREM. *Let χ be a sentence in $\mathcal{L}_{TT_3}$.*

(1) *$NF \vdash \overline{\chi}$ iff $NF_3 + \overline{\mathbf{\Lambda Amb}} \vdash \overline{\chi}$;*
(2) *iff for some k, $TT + \mathbf{\Sigma Amb} \vdash \bigvee\!\!\!\bigvee_{0 \le i \le k} \chi^{i}$.*

[2]Let's remark that the collection of all type raising axioms, $\phi \to \phi^{\dagger}$, generates trivially the usual scheme of ambiguity **Amb** because the collection of sentences is closed under negation. This is not the case for the collection of Σ-sentences. **ΣAmb** is made up of true unidirectional raising axioms.

PROOF. (**a**) If $NF \vdash \overline{\chi}$, then $TT_3 + TT_3^+ + A + \mathbf{Amb}_4 \vdash_4 \chi$, where A is a specific closed axiom of comprehension[3] in $\mathcal{L}_{TT_4}$ of the kind $\exists y^3 \forall x^2 (x^2 \in y^3 \leftrightarrow \phi)$. This combines the results of Specker and Grishin, mentioned above.

(**b**) If $TT_3 + TT_3^+ + A + \mathbf{Amb}_4 \vdash_4 \chi$, then $TT_3 + \mathbf{\Lambda Amb} \vdash_3 \chi$.

Let $\mathcal{M} = \langle M_0, \in_0, M_1, \in_1, M_2 \rangle$ be a model of $TT_3 + \mathbf{\Lambda Amb} + \neg\chi$. If $\phi_1 \dots \phi_n$ are sentences in $\mathcal{L}_{TT_3}$, there is a conjunction $\Psi \equiv (\neg)\phi_1 \wedge \cdots \wedge (\neg)\phi_n$ of these sentences optionally negated that is true in $\mathcal{M}$. By $\mathbf{\Lambda Amb}$, $\mathcal{M} \models (\Psi \wedge \exists u^2 u^2 \models \Psi)$. Let $U \in M_2$ be a witness for $\exists u^2 u^2 \models \Psi$ — we certainly may assume U non-empty because we may include $\exists x^0 x^0 = x^0$ among the ϕ_i.

By Lemma 1, $\mathcal{M}_{\langle U,2 \rangle} \models \Psi^+ \wedge \Psi \wedge A$.

Therefore,

$$\mathcal{M}_{\langle U,2 \rangle} \models TT_3^+ + A + \bigwedge_{1 \le i \le n} (\phi_i \leftrightarrow \phi_i^+) + \neg\chi^+.$$

Compactness produces a model of $TT_3^+ + A + \mathbf{Amb}_4 + \neg\chi^+$, which of course is also a model of $TT_3 + TT_3^+ + A + \mathbf{Amb}_4 + \neg\chi$.

(**c**) If $TT_3 + \mathbf{\Lambda Amb} \vdash_3 \chi$, then, for some k, $TT + \mathbf{\Sigma Amb} \vdash \bigvee_{0 \le i \le k} \chi^i$.

Let's start with a model $\mathcal{M}$ of $TT + \mathbf{\Sigma Amb}$ verifying $\neg\chi, \neg\chi^+, \neg\chi^{++}, \dots$.

Let $\phi_1 \dots \phi_n$ be sentences in $\mathcal{L}_{TT_3}$. There are some p, q, such that $q < p$ and $\mathcal{M} \models \phi_i^q \leftrightarrow \phi_i^p$ for $1 \le i \le n$.

We know, by Lemma 2, that

$$\langle M_q, \in_q, M_{q+1}, \in_{q+1}, M_{q+2}, \in_{q+2}, M_{q+3} \rangle \models (\phi_i \rightarrow \exists u^3 u^3 \models \phi_i).$$

Hence $\mathcal{M} \models (\phi_i^q \rightarrow \exists u^{q+3} u^{q+3} \models \phi_i)$. From this and $\mathbf{\Sigma Amb}$, $\mathcal{M} \models (\phi_i^q \rightarrow \exists u^{p+2} u^{p+2} \models \phi_i)$ because $\exists u^2 u^2 \models \phi_i$ is a Σ-sentence. Hence $(\phi_i^p \rightarrow \exists u^{p+2} u^{p+2} \models \phi_i)$ is true in $\mathcal{M}$.

Therefore,

$$\langle M_p, \in_p, M_{p+1}, \in_{p+1}, M_{p+2} \rangle \models TT_3 + \neg\chi + \bigwedge_{1 \le i \le n} (\phi_i \rightarrow \exists u^2 u^2 \models \phi_i).$$

Compactness gives us a model of $TT_3 + \neg\chi + \mathbf{\Lambda Amb}$.

(a), (b) and (c), dispose of the "if" part of 2. The "only if" part is clear since erasing the types in a derivation in $TT + \mathbf{\Sigma Amb}$ we obtain a derivation in NF because $\overline{\mathbf{\Sigma Amb}}$ is tautologous.

(**d**) If $TT_3 + \mathbf{\Lambda Amb} \vdash_3 \chi$, then $NF_3 + \overline{\mathbf{\Lambda Amb}} \vdash \overline{\chi}$. Again, this is proved by erasing types.

So the "if" part of 1 is completed by (a), (b) and (d).

(**e**) If $NF_3 + \overline{\mathbf{\Lambda Amb}} \vdash \overline{\chi}$, then $NF \vdash \overline{\chi}$. Because $NF \vdash \overline{\mathbf{\Lambda Amb}}$ by Lemma 2. ⊣

COROLLARY. *The theories* NF, $NF_3 + \mathbf{\Lambda Amb}$ *and* $TT + \mathbf{\Sigma Amb}$ *are equiconsistent.*

Comments. (**1**) The first part of the theorem is a description of the 3-stratifiable theorems of NF as a 3-stratifiable extension of NF_3. The 1-stratifiable part (the theory of equality on an infinite domain), 2-stratifiable part and n-stratifiable part

[3][6] uses the axiom stating the existence of the set of sets with non-empty intersection; [3] uses the existence of the set of Leibniz substances (the proof is essentially that of Lemma 2); [1] introduces still another A.

($n > 3$) of NF are respectively identical with the 1, 2-stratifiable part and n-stratifiable part of NF_1, NF_2 (provided NF is consistent!) and NF_n, which is NF itself. On the other hand, it is known that the 3-stratifiable part of NF is *not* the 3-stratifiable part of NF_3 ([2] and [1], [4]).

(**2**) We have been concerned so far with type raising. Let us now consider the converse of **ΣAmb**: $\overleftarrow{\mathbf{\Sigma Amb}}$, i.e., $\phi^+ \to \phi$, for Σ-sentences ϕ. It is easy to derive from the above what is, in our opinion, a less interesting result, namely that $NF \vdash \overline{\chi}$ iff $TT_4 + \overleftarrow{\mathbf{\Sigma Amb}}_4 \vdash_4 \chi$, for 3-typed χ.

This is simply because in $TT_4 + \overleftarrow{\mathbf{\Sigma Amb}}$, one proves **ΛAmb** by using the TT_4-case of Lemma 2.

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