

Reassurance via translation*

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Abstract

Reassurance and classical recapture are scrutinized for minimal versions of non-classical bivalent logics, by embedding non-classical logics in classical logic in order to take advantage of the standard classical background.

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Reassurance and classical recapture will be shown to hold for a minimal version of non-classical logics. This kind of reassurance was first proved by Graham Priest for the logic of paradox for languages with a finite number of relation symbols and no function symbols. Later on he changed his specific definition of minimality, because it appeared to lack recapture. This was not the end of the story, however, because we showed in [1], not only that reassurance fails for his new concept in the languages considered, but also that it is likely to fail for any reasonable notion of minimality when function symbols and equality are included. We then proposed a definition of minimality that fixes the problem for the languages considered by Priest, and left the problem open for languages with predicate and function symbols, but without equality; and for languages containing an infinite number of predicate symbols and equality, but no function symbols.

The aim of this paper is to settle the first problem via a natural translation technique. We will do it not only for LP but for the so called four valued logic and its derived three valued ones. We have checked that the same method can be adapted to solve the second problem, at least if equality is treated in one of the ways suggested in [1].

Though the semantics for these non-classical logics differ from that of standard logic, it is convenient to work in ordinary standard logic and derive the results for non-classical logics from propositions of ordinary classical logic, by translating/embedding the non-classical logic into classical logic. The excess of logical stuff over the image of the translation, will then act as a kind of metalanguage.

The paper is divided into two sections. The first section is a sort of exercise in elementary model theory. The second one transfers/translates the results thereby obtained in the non-classical bivalent logic, which stems mainly from Michael Dunn's semantics in [2].¹

Our main reference for motivation and complementary information is [3], especially chapter 16.

¹See also [3], chapter 5, footnote 3, and the autocommentary of it in chapter 19.

1 Minimal models

Definition 1.1 *Let \mathcal{F} be a set of first-order formulas in a language without equality, but possibly with function symbols.*

The \mathcal{F} -kernel of a model \mathfrak{A} , $\ker_{\mathcal{F}}(\mathfrak{A})$, is the set of the objects o in the universe $|\mathfrak{A}|$ of \mathfrak{A} such that $v(x) = o$ and $(\mathfrak{A}, v) \models A$, for some formula A of \mathcal{F} , some variable x occurring free in A , and some valuation v in $|\mathfrak{A}|$.

Loosely speaking, $\ker_{\mathcal{F}}(\mathfrak{A})$ is the union of the fields of the formulas in \mathcal{F} , viewed as relations on $|\mathfrak{A}|$. If \mathcal{F} is a set of sentences, then the \mathcal{F} -kernel is the emptyset. If a universal closure of some formula in \mathcal{F} with a free variable is a logical truth, then $\ker_{\mathcal{F}}(\mathfrak{A}) = |\mathfrak{A}|$.

The transfer relation $\subset_{\mathcal{F}}$ between models is defined by:

$$\mathfrak{B} \subset_{\mathcal{F}} \mathfrak{A} \quad \text{iff} \quad \begin{array}{l} \ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}) \text{ and} \\ \text{if } (\mathfrak{B}, v) \models A \text{ then, } (\mathfrak{A}, v) \models A, \text{ for all } A \text{ in } \mathcal{F}, \\ \text{and all valuations } v \text{ to } \ker_{\mathcal{F}}(\mathfrak{B}). \end{array}$$

The relation $\prec_{\mathcal{F}}$ is defined by $\mathfrak{B} \subset_{\mathcal{F}} \mathfrak{A}$ and $|\mathfrak{B}| \supseteq |\mathfrak{A}|$.

Note that if \mathcal{F} is the set of all sentences, then $\subset_{\mathcal{F}}$ is the relation of elementary equivalence. Also, if \mathcal{F} is the set of all formulas, and the language has no function symbols, then $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ if and only if \mathfrak{B} is an elementary extension of \mathfrak{A} .

If X is a class of models, an \mathcal{F} -minimal model in X is a model \mathfrak{A} in X such that for every model \mathfrak{B} in X , if $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$, then $\mathfrak{A} \subset_{\mathcal{F}} \mathfrak{B}$.

In particular, if Σ is a theory (a set of sentences), an \mathcal{F} -minimal Σ -model is an \mathcal{F} -minimal model in the class of all models of Σ .

Thus, an \mathcal{F} -minimal model in X is a minimal element of X , relative to the strict partial ordering “ $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ and not $\mathfrak{A} \subset_{\mathcal{F}} \mathfrak{B}$ ”.

Proposition 1.1 *For every \mathcal{F}, Σ , and model \mathfrak{A} of Σ , with finite \mathcal{F} -kernel, there is an \mathcal{F} -minimal Σ -model \mathfrak{B} such that $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$.*

PROOF

We enrich the language by adding a *new* symbol $=$ (that we shall interpret canonically as the identity) and a new constant c_o for each element o of the universe $|\mathfrak{A}|$ of the model \mathfrak{A} of Σ .

The set of these new constants is denoted by \mathcal{A} , and if \mathfrak{C} is a model such that $|\mathfrak{A}| \subseteq |\mathfrak{C}|$, we will denote by $(\mathfrak{C}, \mathcal{A})$ the expansion of \mathfrak{C} for the enriched language, obtained by putting $c_{o(\mathfrak{C}, \mathcal{A})} = o$, for $o \in |\mathfrak{A}|$.

Suppose $\ker_{\mathcal{F}}(\mathfrak{A}) = \{o_1, \dots, o_k\}$ and let $K = \{c_{o_1}, \dots, c_{o_k}\}$. We form a theory Δ , by first adding to Σ all the sentences

- $\neg A[x_1 := \alpha_1, \dots, x_n := \alpha_n]$ that are true in $(\mathfrak{A}, \mathcal{A})$, and where A is in \mathcal{F} , and $\{\alpha_1, \dots, \alpha_n\} \subseteq K$;
- $\forall x_1 \dots \forall x_n (A \rightarrow (x_1 = c_{o_1} \vee \dots \vee x_n = c_{o_k}))$, where $A \in \mathcal{F}$ and x_1 occurs free in A ;²
- $\neg c_o = c_{o'}$, when o, o' are distinct elements of $|\mathfrak{A}|$.

and, next, a maximal set, consistent with the so obtained theory, of sentences $\neg A[x_1 := \alpha_1, \dots, x_n := \alpha_n]$, where A is in \mathcal{F} , and $\{\alpha_1, \dots, \alpha_n\} \subseteq K$.

Let \mathfrak{B} be a model such that $(\mathfrak{B}, \mathcal{A})$ is a model of Δ . We will complete the proof by showing that

1. $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$, and that
2. \mathfrak{B} is an \mathcal{F} -minimal Σ -model.

1. We have $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and $\ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A})$, by the axioms of Δ .

Let A be in \mathcal{F} , and let v be a valuation to $\ker_{\mathcal{F}}(\mathfrak{B})$ such that $(\mathfrak{B}, v) \models A$. We have $(\mathfrak{B}, \mathcal{A}) \models A[x_1 := c_{v(x_1)}, \dots, x_n := c_{v(x_n)}]$. Since $\neg A[x_1 := c_{v(x_1)}, \dots, x_n := c_{v(x_n)}]$ is not in Δ , $(\mathfrak{A}, \mathcal{A}) \not\models \neg A[x_1 := c_{v(x_1)}, \dots, x_n := c_{v(x_n)}]$. Therefore, $(\mathfrak{A}, v) \models A$, because v is a valuation in \mathfrak{A} as well.

2. Let $\mathfrak{B}' \prec_{\mathcal{F}} \mathfrak{B}$ and let us prove that $\mathfrak{B} \subset_{\mathcal{F}} \mathfrak{B}'$.

We first show that $(\mathfrak{B}', \mathcal{A})$ is a model of Δ .

Suppose that $\neg A[x_1 := \alpha_1, \dots, x_n := \alpha_n] \in \Delta$ and that $(\mathfrak{B}', \mathcal{A}) \models A[x_1 := \alpha_1, \dots, x_n := \alpha_n]$, for $A \in \mathcal{F}$ and $\{\alpha_1, \dots, \alpha_n\} \subseteq K$. Then $(\mathfrak{B}', v) \models A$, for a valuation in $\ker_{\mathcal{F}}(\mathfrak{B}')$ such that $v(x_i) = \alpha_{i(\mathfrak{B}', \mathcal{A})}$ ($1 \leq i \leq n$), and it follows that $(\mathfrak{B}, v) \models A$, because $\mathfrak{B}' \prec_{\mathcal{F}} \mathfrak{B}$. Hence, $(\mathfrak{B}, \mathcal{A}) \models A[x_1 := \alpha_1, \dots, x_n :=$

²When the kernel is empty, this is $\forall x_1 \dots \forall x_n \neg A$.

$\alpha_n]$, in contradiction with the fact that $(\mathfrak{B}, \mathcal{A}) \models \Delta$.

The other sentences of Δ are easily seen to be true in $(\mathfrak{B}', \mathcal{A})$.

We finish by showing that $\mathfrak{B} \subset_{\mathcal{F}} \mathfrak{B}'$.

If, for $A \in \mathcal{F}$ and valuation v in $\ker_{\mathcal{F}}(\mathfrak{B}) \subseteq |\mathfrak{B}'|$, $(\mathfrak{B}, v) \models A$ and $(\mathfrak{B}', v) \not\models A$, then $(\mathfrak{B}, \mathcal{A}) \models A[x_1 := c_{v(x_1)}, \dots, x_n := c_{v(x_n)}]$ and $(\mathfrak{B}', \mathcal{A}) \not\models A[x_1 := c_{v(x_1)}, \dots, x_n := c_{v(x_n)}]$. It follows that $\Delta \cup \{\neg A[x_1 := \alpha_1, \dots, x_n := \alpha_n]\}$ would be consistent, since $(\mathfrak{B}', \mathcal{A})$ is a model of it. But this is plainly impossible, by maximality of Δ , and the fact that $(\mathfrak{B}, \mathcal{A})$ is a model of Δ . Therefore, $(\mathfrak{B}', v) \models A$. From this, we also see that $\ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{B}')$. ■

Corollary 1.1 *If \mathcal{F} is a set of sentences and Σ is consistent, then there is a $\prec_{\mathcal{F}}$ -minimal Σ -model.*

PROOF

If \mathcal{F} is a set of sentences, the \mathcal{F} -kernel is empty, and hence finite! ■

1.1 Positive theories

Definition 1.2 *A formula is positive if all its logical symbols are among \wedge, \vee, \forall and \exists .*

A p-trivial theory is one that entails all positive sentences of its language. A model is (positive) trivial iff every positive sentence is true in it.

Note that a model is trivial iff all sentences of the form $\forall x_1 \dots \forall x_n r x_1 \dots x_n$ are true in it.

The next proposition refines the observation that every positive theory has a finite (trivial) model.

Proposition 1.2 *Every non p-trivial positive theory has a finite non-trivial model.*

PROOF

For a simple-minded proof, suppose that \mathfrak{A} is an infinite model such that, for some r^n , $\langle o_1, \dots, o_n \rangle \notin r_{\mathfrak{A}}^n$, and let \spadesuit be an element of $|\mathfrak{A}| \setminus \{o_1, \dots, o_n\}$.

Define a finite model \mathfrak{B} as follows:

$$|\mathfrak{B}| = X \cup \{\spadesuit\};$$

$$f_{\mathfrak{B}}(o_1, \dots, o_n) = \spadesuit, \text{ for all function symbols;}$$

$$r_{\mathfrak{B}}^n = |\mathfrak{B}|^n \setminus \{\langle o_1, \dots, o_n \rangle\};$$

$$s_{\mathfrak{B}}^m = |\mathfrak{B}|^m, \text{ for any other relation symbol } s^m.$$

One shows that every positive sentence true in \mathfrak{A} is true in \mathfrak{B} , by showing, by induction, that, for A positive, $(\mathfrak{B}, v^b) \models A$, if $(\mathfrak{A}, v) \models A$, where $v^b(x) = v(x)$, for $v(x) \in \{o_1, \dots, o_n\}$, and $v^b(x) = \spadesuit$, else.

The basis of the induction is clear and the inductive steps are straightforward, because the set of positive formulas is closed under subformulas. Clearly, $\mathfrak{B} \not\models \forall x_1 \dots \forall x_n r^n x_1 \dots x_n$, that is to say \mathfrak{B} is not trivial. \blacksquare

Definition 1.3 *The consequence relation $\Sigma \Vdash_{\mathcal{F}} C$ holds iff $\mathfrak{A} \models C$, for every \mathcal{F} -minimal Σ -model.³*

Remark *Since an \mathcal{F} -minimal Σ, Π -model ought not be an \mathcal{F} -minimal Σ -model, this consequence relation ought not be monotonic.⁴ For example, $p \Vdash_{\{p,q\}} \neg q$, but $p, q \not\Vdash_{\{p,q\}} \neg q$. Neither is it closed under substitution: $p \Vdash_{\{p,q\}} \neg q$, but $p \not\Vdash_{\{p,q\}} \neg p$. Clearly, if $\Sigma \Vdash C$, then $\Sigma \Vdash_{\mathcal{F}} C$.*

Definition 1.4 *\mathcal{F} transfers triviality between models of a positive theory Σ iff whenever \mathfrak{B} is trivial and $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$, then \mathfrak{A} is trivial, for every model \mathfrak{A} and \mathfrak{B} of Σ .*

It is in the next lemma that the condition on the universes in the definition of $\prec_{\mathcal{F}}$ comes in.

Lemma 1.1 *A sufficient condition for \mathcal{F} to transfer triviality between models of Σ is that for every atomic $rx_1 \dots x_n$ with distinct variables, there are positive formulas A_1, \dots, A_m in \mathcal{F} such that $\Sigma \Vdash (\forall x_1 \dots \forall x_k (A_1 \wedge \dots \wedge A_m) \rightarrow \forall x_1 \dots \forall x_n rx_1 \dots x_n)$ (or $\Sigma \Vdash \forall x_1 \dots \forall x_n rx_1 \dots x_n$).*

³The proper generalisation of this relation to formulas, is achieved by extending the notion of \mathcal{F} -minimal Σ -model to formulas as follows: if \mathcal{G} is a set of formulas, an \mathcal{F} -minimal \mathcal{G} -model is an \mathcal{F} -minimal model in the class $\{\mathfrak{B} \mid (\mathfrak{B}, v) \models \mathcal{G}, \text{ for some } v\}$.

⁴When dealing with sets of sentences, I write Σ, Π for $\Sigma \cup \Pi$ and A for $\{A\}$.

PROOF

Let $\mathfrak{A}, \mathfrak{B}$ be models of Σ such that \mathfrak{B} is trivial and that $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$. Suppose moreover that $\Sigma \Vdash (\forall x_1 \dots \forall x_k (A_1 \wedge \dots \wedge A_m) \rightarrow \forall x_1 \dots \forall x_n r x_1 \dots x_n)$, for some positive formulas A_1, \dots, A_m in \mathcal{F} . Since \mathfrak{B} is trivial, we have $(\mathfrak{B}, v) \models A_i$ ($1 \leq i \leq m$), for every valuation v in $\ker_{\mathcal{F}}(\mathfrak{B}) = |\mathfrak{B}|$. Using the crucial fact that $|\mathfrak{B}| \supseteq |\mathfrak{A}|$, we conclude $(\mathfrak{A}, v) \models A_i$ ($1 \leq i \leq m$), for every valuation in \mathfrak{A} . Therefore $\mathfrak{A} \models \forall x_1 \dots \forall x_n r x_1 \dots x_n$. Since this works for every relational symbol, \mathfrak{A} is trivial. ■

Now we show that one can be reassured with regard to the “triviality” of $\Vdash_{\mathcal{F}}$:

Theorem 1.1 *If \mathcal{F} transfers triviality and the positive theory Σ is not p-trivial, then $\Sigma \not\Vdash_{\mathcal{F}} C$, for some positive C .*

PROOF

Bringing propositions 1.1 and 1.2 together, one sees that every non p-trivial positive theory Σ has a non-trivial \mathcal{F} -minimal Σ -model. ■

In a suitable language, $\Sigma = \emptyset$ and $\mathcal{F} = \{\neg px\}$ provide an example of a non p-trivial positive theory without a non-trivial \mathcal{F} -minimal Σ -model.

We now show that $\Vdash_{\mathcal{F}}$ recaptures the logical consequence \Vdash in some consistent environments.

Theorem 1.2 *Let $\forall \neg \mathcal{F}$ be the universal closures of the negations of the formulas of \mathcal{F} . If $\forall \neg \mathcal{F}, \Sigma$ is consistent, then $\Sigma \Vdash_{\mathcal{F}} C$ iff $\forall \neg \mathcal{F}, \Sigma \Vdash C$.*

PROOF

A model of $\forall \neg \mathcal{F}, \Sigma$ is clearly an \mathcal{F} -minimal Σ -model. For the converse, suppose that \mathfrak{A} is \mathcal{F} -minimal. And let \mathfrak{B} be a model of $\forall \neg \mathcal{F}, \Sigma$. By the upward Löwenheim-Skolem theorem⁵, we can suppose that $|\mathfrak{A}| \subseteq |\mathfrak{B}|$. Therefore $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$. By minimality, \mathfrak{A} is model of $\forall \neg \mathcal{F}$. ■

If $\Sigma = \{\exists x \neg px\}$ and $\mathcal{F} = \{\neg px\}$, we have $\forall \neg \mathcal{F}, \Sigma \Vdash \forall x px$, but not $\Sigma \Vdash_{\mathcal{F}} \forall x px$.

⁵If equality was present and interpreted in a standard way (no externally distinct objects are internally equal), then this argument would not be valid, because it doesn't apply to finite models.

2 Two-valued logic translated in one-valued logic

2.1 Truth / Falsehood

The *bivalent logic* is the natural non-classical logic that emerges from classical logic by simply dropping the principles of excluded-middle and of non-contradiction.⁶ As odd and maybe misleading this use of ‘bivalent’ might sound, it is intended to refer only to the existence of two independent truth-values not to their relations. Hence although classical logic could be introduced as it is often the case along this line as a bivalent consistent and complete logic, it is actually better seen as a univalent partial logic, since the false, there being definable from the truth, need not be considered in it as a primitive value.

A BL-model \mathfrak{A} , with non empty universe $|\mathfrak{A}|$, is exactly like an ordinary model, except that n -ary relation symbols are interpreted by ordered pairs of their respective extension and anti-extension $\langle r_{\mathfrak{A}}^+, r_{\mathfrak{A}}^- \rangle$. Thus, constants and function symbols are interpreted, as usual, by objects and functions. Likewise, a valuation is still a function of the set of the variables; and the valuation v to $|\mathfrak{A}|$ extends canonically to an interpretation $v_{\mathfrak{A}}$ of the terms by putting, $v_{\mathfrak{A}}(x) = v(x)$, for variables x , and $v_{\mathfrak{A}}(ft_1\dots t_m) = f_{\mathfrak{A}}(v_{\mathfrak{A}}(t_1), \dots, v_{\mathfrak{A}}(t_m))$, for complex terms. The truth and falsehood in a model with respect to a valuation are defined inductively, as follows:

$$\begin{aligned}
(\mathfrak{A}, v) \models^+ rt_1\dots t_n & \text{ iff } \langle v_{\mathfrak{A}}(t_1), \dots, v_{\mathfrak{A}}(t_n) \rangle \in r_{\mathfrak{A}}^+ \\
(\mathfrak{A}, v) \models^- rt_1\dots t_n & \text{ iff } \langle v_{\mathfrak{A}}(t_1), \dots, v_{\mathfrak{A}}(t_n) \rangle \in r_{\mathfrak{A}}^- \\
(\mathfrak{A}, v) \models^+ \neg A & \text{ iff } (\mathfrak{A}, v) \models^- A \\
(\mathfrak{A}, v) \models^- \neg A & \text{ iff } (\mathfrak{A}, v) \models^+ A \\
(\mathfrak{A}, v) \models^+ (A \wedge B) & \text{ iff } (\mathfrak{A}, v) \models^+ A \text{ and } (\mathfrak{A}, v) \models^+ B \\
(\mathfrak{A}, v) \models^- (A \wedge B) & \text{ iff } (\mathfrak{A}, v) \models^- A \text{ and/or } (\mathfrak{A}, v) \models^- B \\
(\mathfrak{A}, v) \models^{\pm} (A \vee B) & \text{ iff } (\mathfrak{A}, v) \models^{\pm} \neg(\neg A \wedge \neg B) \\
(\mathfrak{A}, v) \models^{\pm} (A \rightarrow B) & \text{ iff } (\mathfrak{A}, v) \models^{\pm} \neg(A \wedge \neg B)
\end{aligned}$$

⁶See the text accompanying footnote 1.

$$\begin{aligned}
(\mathfrak{A}, v) \models^+ \forall x A & \quad \text{iff} \quad (\mathfrak{A}, v[x \mapsto o]) \models^+ A, \\
& \quad \text{for all } o \text{ in } |\mathfrak{A}| \\
(\mathfrak{A}, v) \models^- \forall x A & \quad \text{iff} \quad (\mathfrak{A}, v[x \mapsto o]) \models^- A, \\
& \quad \text{for some } o \text{ in } |\mathfrak{A}| \\
\text{where } v[x \mapsto o](\alpha) & \text{ is } o, \text{ if } \alpha \text{ is } x, \text{ and else } v(\alpha) \\
(\mathfrak{A}, v) \models^\pm \exists x A & \quad \text{iff} \quad (\mathfrak{A}, v) \models^\pm \neg \forall x \neg A
\end{aligned}$$

We define the *BL-consequence relation* by $\Sigma \Vdash_{\text{BL}} C$ iff for every BL-model \mathfrak{A} and valuation v such that $(\mathfrak{A}, v) \models^+ A$, for A in Σ , we have $(\mathfrak{A}, v) \models^+ C$.

2.2 Positive translation

We will exploit Lyndon's well-known notion of positive and negative occurrence of a relation symbol r in a formula A .

An occurrence of r in A is *positive* when the branch of the parse tree of A leading from this occurrence of r to A itself contains an even number of formulas $\neg F$ or $(F \rightarrow B)$, with the corresponding occurrence of r in F . The occurrence is *negative* when this number of formulas is odd.

A formula is positive [negative] iff all occurrences of relation symbols in it are positive [negative].

Remark *A formula is positive if it can be transformed in a positive formula (in the strict sense of definition 1.2), by replacing $(A \rightarrow B)$ by $(\neg A \vee B)$, pushing negations inside—using the de Morgan and the related quantification laws—and cancelling double negations.*

Let us enrich our language \mathcal{L} to a language \mathcal{L}^{pos} by adding a new symbol \bar{r} for each relational symbol r in \mathcal{L} . If we replace in a formula A of \mathcal{L} each occurrence of an atomic formula $rt_1 \dots t_n$, with negative occurrence of r , by $\neg \bar{r}t_1 \dots t_n$, then we obtain a positive formula A^{pos} in \mathcal{L}^{pos} . If we replace similarly each occurrence of an atomic formula $rt_1 \dots t_n$, with positive occurrence of r , by $\bar{r}t_1 \dots t_n$, then we obtain a negative formula A^{neg} .⁷ We denote by

⁷To make this formal, define first $rt_1 \dots t_n^{\text{pos}}$ as $rt_1 \dots t_n$ and $rt_1 \dots t_n^{\text{neg}}$ as $\neg \bar{r}t_1 \dots t_n$. And then define A^{pos} and A^{neg} inductively as follows: $\neg A^\alpha$ is $\neg A^{\bar{\alpha}}$, $(A \wedge B)^\alpha$ is $(A^\alpha \wedge B^\alpha)$, $(A \vee B)^\alpha$ is $(A^\alpha \vee B^\alpha)$, $(A \rightarrow B)^\alpha$ is $(A^{\bar{\alpha}} \rightarrow B^\alpha)$, $\forall x A^\alpha$ is $\forall x A^\alpha$, and $\exists x A^\alpha$ is $\exists x A^\alpha$; where α stands for pos or for neg and $\bar{\text{pos}}$ is neg , and $\bar{\text{neg}}$ is pos .

\mathcal{T}^{pos} the set of A^{pos} such that A is in \mathcal{T} .

To a BL-model \mathfrak{A} , we associate in a biunivocal way a model $\mathfrak{A}^{\text{pos}}$ for \mathcal{L}^{pos} , with the same universe and interpretation of the function symbols, by putting:

$$\begin{aligned} r_{\mathfrak{A}^{\text{pos}}} &= r_{\mathfrak{A}}^+ \\ \bar{r}_{\mathfrak{A}^{\text{pos}}} &= r_{\mathfrak{A}}^-, \end{aligned}$$

for all relation symbols in the language.

An easy induction on t and A gives:

Lemma 2.1

$$\begin{aligned} v_{\mathfrak{A}}(t) &= v_{\mathfrak{A}^{\text{pos}}}(t); \\ (\mathfrak{A}, v) \models^+ A &\quad \text{iff} \quad (\mathfrak{A}^{\text{pos}}, v) \models A^{\text{pos}}; \\ (\mathfrak{A}, v) \models^- A &\quad \text{iff} \quad (\mathfrak{A}^{\text{pos}}, v) \not\models A^{\text{neg}}. \end{aligned}$$

Thus the positive translation A^{pos} expresses that A is true, and the negative translation A^{neg} that A is not false.

The next proposition follows as a corollary.

Proposition 2.1 $\Sigma \Vdash_{\text{BL}} C$ iff $\Sigma^{\text{pos}} \Vdash C^{\text{pos}}$.

2.3 Gaps and Gluts

2.3.1 The general bivalent case

Definition 2.1 The non-classical part of the bivalent model \mathfrak{A} is defined à la Priest as the set of contradictory or incomplete statements in \mathfrak{A} , namely the set $\mathfrak{A}!$

$$\left\{ \langle r^n, \langle o_1, \dots, o_n \rangle \rangle \mid \begin{array}{l} r^n \text{ is a relation symbol; and} \\ \langle o_1, \dots, o_n \rangle \in r_{\mathfrak{A}}^{n+} \cap r_{\mathfrak{A}}^{n-} \text{ or } \langle o_1, \dots, o_n \rangle \notin r_{\mathfrak{A}}^{n+} \cup r_{\mathfrak{A}}^{n-} \end{array} \right\}$$

The preorder⁸ \prec is defined as

$$\mathfrak{B} \prec \mathfrak{A} \quad \text{iff} \quad \mathfrak{B}! \subseteq \mathfrak{A}! \text{ and } |\mathfrak{B}| \supseteq |\mathfrak{A}|$$

⁸Priest uses strict inclusion between the $\mathfrak{C}!$, and so has a strict partial order. This change does not affect the definition of minimality.

If Σ is a set of sentences, a minimal Σ -model \mathfrak{A} is a model of Σ such that if $\mathfrak{B} \models \Sigma$ and $\mathfrak{B} \prec \mathfrak{A}$, then $\mathfrak{A}! \subseteq \mathfrak{B}!$.

The associated consequence relation $\Sigma \Vdash_{\text{BL}^m} C$ is defined as $\mathfrak{A} \models^+ C$, for every Σ -minimal model \mathfrak{A} such that $\mathfrak{A} \models^+ \Sigma$.

Definition 2.2 We denote by NC the set of all formulas $(rx_1 \dots x_n \wedge \neg rx_1 \dots x_n)$ of the language; and by EM the set of all sentences $\forall x_1 \dots \forall x_n (rx_1 \dots x_n \vee \neg rx_1 \dots x_n)$ of the language, where x_1, \dots, x_n are the first n variables in some fixed enumeration of the variables.

Proposition 2.2

Let \mathcal{F} be the set of all formulas in NC^{pos} or NC^{neg} , then

1. $\Sigma \Vdash_{\text{BL}^m} C$ iff $\Sigma^{\text{pos}} \Vdash_{\mathcal{F}} C^{\text{pos}}$.
2. \mathcal{F} transfers triviality between models of Σ^{pos} .

PROOF

1. is true if it is the case that a BL -model \mathfrak{A} is a minimal Σ -model iff $\mathfrak{A}^{\text{pos}}$ is an \mathcal{F} -minimal Σ^{pos} -model. And this follows immediately from the fact that

$$\mathfrak{B}! \subseteq \mathfrak{A}! \quad \text{iff} \quad \mathfrak{B}^{\text{pos}} \subset_{\mathcal{F}} \mathfrak{A}^{\text{pos}}$$

which we now prove.

Suppose that $\mathfrak{B}! \subseteq \mathfrak{A}!$, and let v be a valuation in $\ker_{\mathcal{F}}(\mathfrak{B}^{\text{pos}})$, such that $(\mathfrak{B}^{\text{pos}}, v) \models (rx_1 \dots x_n \wedge \neg \bar{r}x_1 \dots x_n)$ or $(\mathfrak{B}^{\text{pos}}, v) \models (\bar{r}x_1 \dots x_n \wedge \neg rx_1 \dots x_n)$. Then, $\langle r, \langle v(x_1), \dots, v(x_n) \rangle \rangle \in \mathfrak{B}! \subseteq \mathfrak{A}!$. Therefore, $(\mathfrak{A}^{\text{pos}}, v) \models (rx_1 \dots x_n \wedge \neg \bar{r}x_1 \dots x_n)$ or $(\mathfrak{A}^{\text{pos}}, v) \models (\bar{r}x_1 \dots x_n \wedge \neg rx_1 \dots x_n)$. This shows also that $\ker_{\mathcal{F}}(\mathfrak{B}^{\text{pos}}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}^{\text{pos}})$.

For the converse, suppose that $\mathfrak{B}^{\text{pos}} \subset_{\mathcal{F}} \mathfrak{A}^{\text{pos}}$. Let $\langle r, \langle o_1, \dots, o_n \rangle \rangle$ be in $\mathfrak{B}!$ and v be a valuation such that $v(x_1) = o_1, \dots, v(x_n) = o_n$. As we have $\langle o_1, \dots, o_n \rangle \in r_{\mathfrak{B}}^+ \cap r_{\mathfrak{B}}^-$ or $\langle o_1, \dots, o_n \rangle \notin r_{\mathfrak{B}}^+ \cup r_{\mathfrak{B}}^-$, it follows that $(\mathfrak{B}^{\text{pos}}, v) \models (rx_1 \dots x_n \wedge \neg \bar{r}x_1 \dots x_n)$ or $(\mathfrak{B}^{\text{pos}}, v) \models (\bar{r}x_1 \dots x_n \wedge \neg rx_1 \dots x_n)$. Therefore, $(\mathfrak{A}^{\text{pos}}, v) \models (rx_1 \dots x_n \wedge \neg \bar{r}x_1 \dots x_n)$ or $(\mathfrak{A}^{\text{pos}}, v) \models (\bar{r}x_1 \dots x_n \wedge \neg rx_1 \dots x_n)$ and whence $\langle o_1, \dots, o_n \rangle \in r_{\mathfrak{A}}^+ \cap r_{\mathfrak{A}}^-$ or $\langle o_1, \dots, o_n \rangle \notin r_{\mathfrak{A}}^+ \cup r_{\mathfrak{A}}^-$, i.e. $\langle r, \langle o_1, \dots, o_n \rangle \rangle \in \mathfrak{A}!$

2. Since $\forall x_1 \dots \forall x_n (rx_1 \dots x_n \wedge \bar{r}x_1 \dots x_n) \rightarrow \forall x_1 \dots \forall x_n rx_1 \dots x_n$ and $\forall x_1 \dots \forall x_n (rx_1 \dots x_n \wedge \bar{r}x_1 \dots x_n) \rightarrow \forall x_1 \dots \forall x_n \bar{r}x_1 \dots x_n$ are logical truths, \mathcal{F} transfers triviality between models of Σ^{pos} , by lemma 1.1. \blacksquare

Definition 2.3 A BL-model is trivial iff every sentence is true in it. Clearly, a BL-model \mathfrak{A} is trivial iff $r_{\mathfrak{A}}^{n+} = r_{\mathfrak{A}}^{n-} = |\mathfrak{A}|^n$, for every r^n .

A theory Σ is BL-trivial [BL^m-trivial] iff $\Sigma \Vdash_{\text{BL}} C$ [$\Sigma \Vdash_{\text{BL}^m} C$], for every C in its language.

Theorem 2.1 (Reassurance) If a theory is not BL-trivial, then it is not BL^m-trivial either.

PROOF

If Σ is not BL-trivial, then Σ^{pos} , is not p-trivial, by proposition 2.1. By proposition 2.2.2 and theorem 1.1, we then have $\Sigma^{\text{pos}} \not\Vdash_{\text{NC}^{\text{pos}}, \text{NC}^{\text{neg}}} C^{\text{pos}}$, for some C . Therefore, by proposition 2.2.1, $\Sigma \not\Vdash_{\text{BL}^m} C$, for some C . ■

2.3.2 The glut and gap cases

Definition 2.4 An LP [K3] model \mathfrak{A} is a BL-model satisfying $r_{\mathfrak{A}}^{n+} \cup r_{\mathfrak{A}}^{n-} = |\mathfrak{A}|^n$ [$r_{\mathfrak{A}}^{n+} \cap r_{\mathfrak{A}}^{n-} = \emptyset$], for every relation symbol r^n in its language.

We observe that there are trivial LP-models, but that no K3-model is trivial.

A CL-model is a model that is both an LP-model and a K3-model.⁹

An LP-model is a BL-model with no gaps. A K3-model is a BL-model with no gluts. Thus an LP-model is a BL-model of EM, and a K3-model is a BL-model in which no sentence of EM is false.

With the consequences relations with respect to LP, K3 and CL-models (\Vdash_{LP} , \Vdash_{K3} and \Vdash_{CL}) we have:

$$\Sigma \Vdash_{\text{BL}} C \begin{array}{l} \rightrightarrows \\ \searrow \end{array} \begin{array}{l} \Sigma \Vdash_{\text{LP}} C \\ \Sigma \Vdash_{\text{K3}} C \end{array} \begin{array}{l} \rightrightarrows \\ \searrow \end{array} \Sigma \Vdash_{\text{CL}} C \iff \Sigma \Vdash C$$

⁹LP stands for the logic of paradox and K3 stands for the gappy Kleene's logic. CL is the BL-version of classical logic, as a classical (univalent) model can be canonically identified with a bivalent model without gaps and gluts.

A BL-model \mathfrak{A} is an LP-model iff $\mathfrak{A}^{\text{pos}}$ is a model of EM^{pos} , i.e. a model verifying all the sentences $\forall x_1 \dots \forall x_n (r^n x_1 \dots x_n \vee \bar{r}^n x_1 \dots x_n)$; and it is a K3-model iff $\mathfrak{A}^{\text{pos}}$ is a model of EM^{neg} , i.e. a model verifying all the sentences $\forall x_1 \dots \forall x_n \neg(r^n x_1 \dots x_n \wedge \bar{r}^n x_1 \dots x_n)$.

We also have the glut and gap versions of proposition 2.1:

Proposition 2.3

1. $\Sigma \Vdash_{\text{LP}} C$ iff $\text{EM}^{\text{pos}}, \Sigma^{\text{pos}} \Vdash C^{\text{pos}}$;
2. $\Sigma \Vdash_{\text{K3}} C$ iff $\text{EM}^{\text{neg}}, \Sigma^{\text{pos}} \Vdash C^{\text{pos}}$;
3. $\Sigma \Vdash C$ iff $\Sigma \Vdash_{\text{CL}} C$ iff $\text{EM}^{\text{pos}}, \text{EM}^{\text{neg}}, \Sigma^{\text{pos}} \Vdash C^{\text{pos}}$.

PROOF

We have $\Sigma \Vdash_{\text{LP}} C$ iff $\text{EM}, \Sigma \Vdash_{\text{BL}} C$. From this, we get 1, by proposition 2.1.

Let E_1, \dots, E_n be the sentences of EM whose relation symbols occur in Σ, C .

We have $\Sigma \Vdash_{\text{K3}} C$ iff $\Sigma \Vdash_{\text{BL}} C \vee \neg E_1 \vee \dots \vee \neg E_n$. From this, we get, by proposition 2.1, $\Sigma \Vdash_{\text{K3}} C$ iff $\Sigma^{\text{pos}} \Vdash C^{\text{pos}} \vee \neg E_1^{\text{neg}} \vee \dots \vee \neg E_n^{\text{neg}}$, whence 2.

Finally, $\Sigma \Vdash_{\text{CL}} C$ iff $\Sigma, \text{EM} \Vdash_{\text{BL}} C \vee \neg E_1 \vee \dots \vee \neg E_n$. From this 3. follows similarly, by proposition 2.1. ■

Theorem 2.2 (Classical recapture for BL^m) *If Σ is consistent, then $\Sigma \Vdash C$ iff $\Sigma \Vdash_{\text{BL}^m} C$.*

PROOF

The set $\forall \neg \mathcal{F}$ of the universal closures of the negations of the formulas of $\text{NC}^{\text{pos}}, \text{NC}^{\text{neg}}$ is equivalent to $\text{EM}^{\text{pos}}, \text{EM}^{\text{neg}}$. By proposition 2.3.3, $\forall \neg \mathcal{F}, \Sigma^{\text{pos}}$ is consistent. Hence, by theorem 1.2, $\forall \neg \mathcal{F}, \Sigma^{\text{pos}} \Vdash C^{\text{pos}}$ iff $\Sigma^{\text{pos}} \Vdash_{\mathcal{F}} C^{\text{pos}}$. From this, the result follows, by propositions 2.2.1 and 2.3.3. ■

Reassurance and recapture Reassurance and classical recapture hold for the gap and for the glut cases, but their respective significance is quite different, as we shall now show.

Definition 2.5 A minimal Σ -LP-model is a minimal Σ -BL-model, which is an LP-model; likewise, a minimal Σ -K3-model is a minimal Σ -BL-model, which is a K3-model.

Notice that if $\mathfrak{A}, \mathfrak{B}$ are Σ -BL-models and \mathfrak{A} is an LP-model [K3-model] such that $\mathfrak{B} \prec \mathfrak{A}$, then \mathfrak{B} is an LP-model [K3-model]. Therefore, a minimal Σ -LP-model [Σ -K3-model] is a minimal Σ -BL-model, for the relation \prec restricted to LP-models [K3-models] of Σ ; and conversely.

To a model \mathfrak{A} for \mathcal{L}^{pos} , we associate a “glut-model” $\mathfrak{A}^{\text{glut}}$ and a “gap-model” $\mathfrak{A}^{\text{gap}}$, each of them with same universe and same interpretation of constants and of function symbols as \mathfrak{A} , by stipulating:

$$\begin{aligned} r_{\mathfrak{A}^{\text{glut}}}^n &= r_{\mathfrak{A}}^n; & r_{\mathfrak{A}^{\text{gap}}}^n &= r_{\mathfrak{A}}^n; \\ \bar{r}_{\mathfrak{A}^{\text{glut}}}^n &= \bar{r}_{\mathfrak{A}}^n \cup (|\mathfrak{A}|^n \setminus r_{\mathfrak{A}}^n); & \bar{r}_{\mathfrak{A}^{\text{gap}}}^n &= \bar{r}_{\mathfrak{A}}^n \setminus r_{\mathfrak{A}}^n, \end{aligned}$$

for all r^n .

One shows, by induction on A , that:

$$\begin{aligned} \text{if } \mathfrak{A} \models A^{\text{pos}}, \quad \text{then } \mathfrak{A}^{\text{glut}} \models A^{\text{pos}}; \\ \text{if } \mathfrak{A}^{\text{gap}} \models A^{\text{pos}}, \quad \text{then } \mathfrak{A} \models A^{\text{pos}}. \end{aligned}$$

Proposition 2.4

1. Every minimal Σ -BL-model is an LP-model .
2. Every minimal Σ -K3-model is a CL-model (hence, an LP-model as well).

PROOF

Let \mathcal{F} be, as above, the set of all formulas $\text{NC}^{\text{pos}} \cup \text{NC}^{\text{neg}}$.

1. Let \mathfrak{A} be a minimal Σ -model. We have $\mathfrak{A}^{\text{posglut}} \prec_{\mathcal{F}} \mathfrak{A}^{\text{pos}}$, because, in the first place, $(\mathfrak{A}^{\text{posglut}}, v) \models (rx_1 \dots x_n \wedge \bar{r}x_1 \dots x_n)$ implies $\langle v(x_1), \dots, v(x_n) \rangle \in r_{\mathfrak{A}^{\text{pos}}} \cap (\bar{r}_{\mathfrak{A}^{\text{pos}}} \cup (|\mathfrak{A}^{\text{pos}}|^n \setminus r_{\mathfrak{A}^{\text{pos}}})) = r_{\mathfrak{A}^{\text{pos}}} \cap \bar{r}_{\mathfrak{A}^{\text{pos}}}$, hence $(\mathfrak{A}^{\text{pos}}, v) \models (rx_1 \dots x_n \wedge \bar{r}x_1 \dots x_n)$; and, in the second place, $(\mathfrak{A}^{\text{posglut}}, v) \models (\neg rx_1 \dots x_n \wedge \neg \bar{r}x_1 \dots x_n)$ implies trivially $(\mathfrak{A}^{\text{pos}}, v) \models (\neg rx_1 \dots x_n \wedge \neg \bar{r}x_1 \dots x_n)$. As a consequence, $\ker_{\mathcal{F}}(\mathfrak{A}^{\text{posglut}}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}^{\text{pos}})$.

Since, by proposition 2.2, $\mathfrak{A}^{\text{pos}}$ is an \mathcal{F} -minimal Σ^{pos} -model, it follows that $\mathfrak{A}^{\text{pos}} \subset_{\mathcal{F}} \mathfrak{A}^{\text{posglut}}$. Hence \mathfrak{A} is an LP-model.

2. A minimal Σ -K3-model is an LP-model by 1, hence a CL-model. ■

For the consequence relations \Vdash_{LP^m} and \Vdash_{K3^m} , and the associated notions of triviality, defined in the obvious way, we obtain

Corollary 2.1

The consequence relation \Vdash_{BL^m} is extensionally the same as \Vdash_{LP^m} .

The consequence relations \Vdash_{K3^m} , \Vdash_{CL} and \Vdash are extensionally identical.

Whence, by the theorems 1.1 and 1.2:

Theorem 2.3 (Reassurance and Recapture)

1. If Σ is not LP-trivial, then it is not LP^m -trivial; if Σ is not K3-trivial, then it is not K3^m -trivial.

2. If Σ is consistent, then

$\Sigma \Vdash_{\text{BL}^m} C$ iff $\Sigma \Vdash_{\text{LP}^m} C$ iff $\Sigma \Vdash_{\text{K3}^m} C$ iff $\Sigma \Vdash C$.

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