112
Notre Dame Journal of Formal Logic
Volume 33, Number 1, Winter 1992

On *NFU* *

MARCEL CRABBÉ

Abstract We first describe a general method for building models of *NFU*, i.e. *NF* with atoms, without the help of Ramsey's theorem. Then, we show that *NFU* has essentially the same strength as *NF* without the axiom of extensionality. Further, we consider axioms stating that there is a set extensionally equivalent to every nonextensional object. By using a variant of the technique of permutations, we prove the consistency of such axioms of weak extensionality with the stratifiable axioms of comprehension. Finally, we show that there is an axiom of weak extensionality that implies full extensionality.

Introduction The stratifiable axioms of comprehension form a system of set theory called *SF*, for 'Stratified Foundations'. In this system, it is possible to deal with various different objects which can contain, nevertheless, the same elements. Such objects are said to be extensionally equivalent.

An object that is not extensionally equivalent to another one is called extensional. The axiom of extensionality says that everything is extensional. Quine's *New Foundations*, *NF*, is *SF* plus the axiom of extensionality. A nonextensional object can be thought of as a *concept*, e.g. the concepts 'founder of *NF*' and 'author of *Word and Object*' are not identical, though their extension is the same unit set. Accordingly, a set theory lacking full extensionality can be viewed as a theory of concepts.

An axiom of weak extensionality states that only some objects are extensional. The weak axiom of extensionality of *NFU* is the statement that the nonextensional objects are all empty, i.e. *Urelemente* or empty concepts.

The problem of the consistency of *NF* is still left open, but the consistency of *NFU*, thus of *SF*, was shown in [4] by Jensen (see also Boffa [1] and [2]). Al-

*A first version of the paper was presented at the *NF* meeting, held at the *Mathematisches Forschungsinstitut Oberwolfach* in March 1987.

Received December 3, 1990; revised March 27, 1991

though *NFU* seems to be, at first sight, nothing less than a "slight (?) modification of *NF*" as Jensen called it, his result proved quite the opposite. This is because it uses only elementary means, whereas, as is well known, higher-order arithmetic is interpretable in *NF*. Jensen's idea is roughly the following: in the form of a corollary of a result given by Specker [7], a model of *NFU* can be obtained from a model of type theory with Urelemente, in which the levels are somehow indiscernible. Taking this fact into account, one can use Ramsey's theorem in order to show that such a model can be constructed from any model of *TT*.

Incidentally, it should be noted that *NF* was designed by Quine (see [5]) in order to solve Russell's paradox, without imposing the draconian measures Russell took in introducing his theory of types. According to Quine's diagnosis, the problem of the paradox lies in the fact that the defining formula of the paradoxical set is unstratifiable. Extensionality plays as little a role in the motivation behind the proposal of *NF* as it does in the paradox itself. So the fact that *SF* is consistent suffices to justify Quine's claim.

Amazingly, the addition of the axiom of extensionality to the axioms of comprehension of *SF* gives rise to an entirely new situation, because one becomes able to disprove the axiom of choice (see Specker [6] and Crabbé [3]). The study of *NF* is therefore a topic quite distinct from the one which concerns the solution of the paradox *stricto sensu*.

After a few basic definitions, this paper is divided into two independent parts. In the first one, we describe a general technique that enables one to obtain models of *SF* (and of *NFU*) without invoking Ramsey's theorem. This requires a *detour* via type theory and models.

In the second part, we will be concerned with first-order language and syntax only. We first show that *NFU* has essentially the same strength as *SF*; by providing an interpretation of *NFU* in *SF*. Then, we shall consider axioms of weak extensionality, stating that there is an object extensionally equivalent to every non-extensional object: all concepts have the same extension. The axiom of extensionality of *NFU* is of this kind. We will use a variant of the familiar technique of permutations in order to prove the consistency of *SF* with such axioms of weak extensionality. The final result is that there is an axiom of weak extensionality that implies full extensionality.

SF, NFU, NF and TT $\mathcal{L}_{NF}$, the language of *NF*, is the same as the language of *SF* (Stratifiable Foundations), *NFU*, and *ZF*, that is first-order predicate calculus, with $=$ and $\in$.

To each variable x of the $\mathcal{L}_{NF}$, and each natural number i, we let correspond a new variable x^i, of *type i*. These new variables are the variables of the language of type theory: $\mathcal{L}_{TT}$. The formulas of $\mathcal{L}_{TT}$ are built up from atomic formulas of the form $x^i \in y^{i+1}$ and $x^i = y^i$, as usual.

If σ is a function from the set of first-order variables of $\mathcal{L}_{NF}$, into the natural numbers, and φ is a first-order formula, φ^σ is the expression resulting from the substitution of $x^{\sigma(x)}$ for x in φ, for each variable x. σ is a *stratification assignment* for the first-order formula φ iff φ^σ is a formula of $\mathcal{L}_{TT}$. φ is *stratifiable* iff there is a stratification assignment for (some alphabetic variant of) φ.

An *axiom of comprehension* in $\mathcal{L}_{NF}$ is a stratifiable formula of the form

$$\exists y \forall x (x \in y \leftrightarrow \varphi),$$

where y is not free in φ.

We write $x \sim y$ for $\forall z (z \in x \leftrightarrow z \in y)$, and $Ext(x)$ for $\forall y (x \sim y \rightarrow x = y)$. $x \sim y$ and $Ext(x)$ can be read as 'x is extensionally equivalent to y' and 'x is extensional', respectively.

The *axiom of extensionality* in $\mathcal{L}_{NF}$ is the sentence:

$$\forall x \, Ext(x).$$

The *axiom of extensionality for nonempty sets* in $\mathcal{L}_{NF}$ is the sentence:

$$\forall x (\exists v \, v \in x \rightarrow Ext(x)).$$

The nonlogical axioms of *SF* are the axioms of comprehension. *NFU* is *SF* plus the axiom of extensionality for nonempty sets. *NF* is *SF* plus the axiom of extensionality.

The logic of *TT* is the usual multisorted logic. The nonlogical axioms of *TT* are the formulas φ^σ, where φ is an axiom of comprehension, or of extensionality in $\mathcal{L}_{NF}$, and σ is a stratification assignment for φ.

1 The consistency of SF* and *NFU Henceforth, a relation will always be a binary relation, i.e. a set of ordered pairs. A *typed structure*, $\mathcal{M}$, is a pair $\langle\langle M_0, M_1, \ldots \rangle, R\rangle$, where $\langle M_0, M_1, \ldots \rangle$ is a denumerable sequence of nonempty sets, and R is a relation.

A valuation v into a typed structure, $\mathcal{M}$, is a function from the variables of $\mathcal{L}_{TT}$ into $\bigcup_{i\in\omega} M_i$ such that $v(x^i) \in M_i$, for each i. $\mathcal{M}, v \models \varphi$ is defined in the usual way starting from the initial clauses:

$$\mathcal{M}, v \models x^i \in y^{i+1} \quad \text{iff } v(x^i) R v(y^{i+1})$$

$$\mathcal{M}, v \models x^i = y^i \quad \text{iff } v(x^i) = v(y^i).$$

We write $\mathcal{M} \models \varphi$ iff $\mathcal{M}, v \models \varphi$ for every valuation v, and $\mathcal{M} \models TT$ iff $\mathcal{M} \models \varphi$, for every axiom φ of *TT*.

Clearly, if $\mathcal{M} = \langle M, R\rangle$, is a first-order structure, and σ a stratification assignment for the first-order formula φ, then:

$$\langle\langle M, M, \ldots \rangle, R\rangle \models \varphi^\sigma \quad \text{iff } \langle M, R\rangle \models \varphi.$$

Let $\mathcal{M} = \langle\langle M_0, M_1, \ldots \rangle, R\rangle$, and $\mathcal{N} = \langle\langle N_0, N_1, \ldots \rangle, S\rangle$. An *isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ is a sequence of functions $\langle \pi_0, \pi_1 \ldots \rangle$ such that π_i is a bijective mapping from M_i to N_i and such that aRb iff $\pi_i(a) S \pi_{i+1}(b)$, for $a \in M_i$ and $b \in M_{i+1}$.

Collapsing lemma *If, for every $i > 0$, $\mathcal{M} = \langle\langle M_0, M_1, \ldots \rangle, R\rangle \models \forall x^i Ext(x^i)$, then $\mathcal{M}$ is isomorphic to a typed structure $\langle\langle M'_0, M'_1, \ldots \rangle, \in\rangle$ such that $M_0 = M'_0$ and $M'_{i+1} \subseteq \mathcal{P}(M'_i)$.*

A structure of this kind is called $\in$-*standard*. If A and S are sets, and if R is a relation, then A is *R-coded* in S iff there is an element $a \in S$ such that

$\forall x(x \in A \leftrightarrow xRa)$. For example, A is $\in$-coded in S iff $A \in S$. Of course, it is possible for a set to be R-coded in different ways.

Let N be a subset of M_0. Then the *natural structure* over N in $\mathcal{M}$, $\mathrm{Nat}_{\mathcal{M}}(N)$, is the ordered pair $\langle\langle N_0, N_1, \ldots\rangle, R\rangle$, where, $N_0 = N$, and, for $i \geq 0$, $N_{i+1} = \{x \in M_{i+1} \mid (\forall y \in M_i)(yRx \to y \in N_i)\}$. $\mathcal{M}^+$ is the typed structure arising from $\mathcal{M}$ by deleting M_0, i.e. $\mathcal{M}^+ = \langle\langle M_1, M_2, \ldots\rangle, R\rangle$.

Theorem *Let $\mathcal{M} \models TT$, N a subset of M_0, R-coded in M_1, and $\langle\delta_0, \delta_1, \ldots\rangle$ an isomorphism from $\mathcal{M}^+$ to $\mathrm{Nat}_{\mathcal{M}}(N)$. Furthermore, let $\in_\delta$ be the relation on N such that $a \in_\delta b$ iff $a \in \delta_0^{-1}(b)$, for a, b in N. Then, $\langle N, \in_\delta\rangle$ is a model of SF.*

Proof: Using the collapsing lemma, we can suppose that $\mathcal{M}$ is $\in$-standard. Therefore, $\mathcal{N}$ is also $\in$-standard and if $a \subseteq N_k$ and $a \in M_{k+1}$, then $a \in N_{k+1}$.

We define, by induction, the elements $\lambda^k(a)$ of N_k for natural numbers k and elements a of N: $\lambda^0(a) = a$, $\lambda^{i+1}(a) = \delta_{i+1}(\{x \in M_{i+1} \mid \lambda^i(a) \in x\})$.

Thus, for each k, $L_k = \{\lambda^k(x) \mid x \in N\}$ is a subset of N_k, and the following hold, for any $a, b \in N$:

$a \in \lambda^1(b)$ iff $b \in_\delta a$,
$\lambda^{k+1}(a) \in \lambda^{k+2}(b)$ iff $\lambda^k(a) \in \lambda^{k+1}(b)$,
if $\lambda^{k+1}(a) = \lambda^{k+1}(b)$, then $\lambda^k(a) = \lambda^k(b)$.

If R is a relation, define $aR^{op}b$ as bRa. If φ is a formula of $\mathcal{L}_{NF}$, φ^{op} is the formula that results from the replacement of the atomic subformulas $x \in y$ by $y \in x$. Clearly, φ^{op} is stratifiable iff φ is.

We see that $\langle\lambda^0, \lambda^1, \ldots\rangle$ is an isomorphism between the typed structures $\langle\langle N, N, \ldots\rangle, \in_\delta^{op}\rangle$ and $\langle\langle L_0, L_1, \ldots\rangle, \in\rangle$. Therefore, if σ is a stratification assignment for φ^{op}, then $\langle N, \in_\delta\rangle \models \varphi$ iff $\langle\langle L_0, L_1, \ldots\rangle, \in\rangle \models \varphi^{op^\sigma}$. Thus, it remains to be shown that $\langle\langle L_0, \ldots, L_1, \ldots\rangle, \in\rangle \models \varphi^{op^\sigma}$, for every axiom of comprehension, φ of *SF*.

Since $\mathcal{M} \models TT$, this will follow from the two following facts:

(1) $L_k \in N_{k+1}$, for $k \geq 0$
(2) If $a \subseteq L_{k+1}$ and $a \in M_{k+2}$, then a is $\in^{op}$-coded in L_k, for $k \geq 0$.

First, we observe that $\{\delta_k(x) \mid x \in M_{k+1} \wedge x \in a\} = \delta_{k+1}(a)$, for $a \in M_{k+2}$. (1) is then proved by induction as follows:

$$L_0 = N \in N_1; \quad L_{k+1} = \{\delta_{k+1}(\{y \in M_{k+1} \mid x \in y\}) \mid x \in L_k\}$$
$$= \delta_{k+2}(\{\{y \in M_{k+1} \mid x \in y\} \mid x \in L_k\}) \in N_{k+2}.$$

Concerning (2), we note first that if $a \in N_{k+2}$, then $\{x \in N \mid \lambda^{k+1}(x) \in a\} = \{x \in N \mid \lambda^k(x) \in \{z \in M_k \mid \{y \in M_{k+1} \mid z \in y\} \in \delta_{k+2}^{-1}(a)\}\}$. Thus, by induction, for any $a \in N_{k+2}$, $\{x \in N \mid \lambda^{k+1}(x) \in a\} \in N_1$. Therefore, $\lambda^k(\delta_0(\{x \in N \mid \lambda^{k+1}(x) \in a\})) \in^{op}$-codes a in L_k, for any a such that $a \subseteq L_{k+1}$, and $a \in M_{k+2}$.

Corollary *Given the assumptions of the theorem, let S be $\delta_0(\delta_1(N_1))$ and define the relation $\in'_\delta$ on N as follows: $a \in'_\delta b$ iff $a \in_\delta b$ and $b \in_\delta S$. Then $\langle N, \in'_\delta\rangle$ is a model of NFU.*

Proof: Since we will soon interpret *NFU* in *SF*, we will not write the proof in full detail, but give only the hints.

For each a in N, there exists a b, namely $\delta_0(\delta_0^{-1}(a) \cap N)$, such that $b \in_\delta S$ and $(\forall x \in N)(x \in_\delta b \leftrightarrow x \in_\delta a)$.

If $a \in_\delta S$, $b \in_\delta S$, and $(\forall x \in N)(x \in'_\delta a \leftrightarrow x \in'_\delta b)$, then $a = b$.

We conclude this first part by showing how to use the theorem in order to construct models of *SF* or *NFU*. If one has a model $\mathcal{M}$ of *TT* and a nonempty subset N of M_0 coded in M_1 such that $\mathcal{M}^+$ and $\mathrm{Nat}_{\mathcal{M}}(N)$ are elementarily equivalent, then one can use Specker's theorem (see [7]) in order to fulfill the hypotheses of the theorem. The existence of such models of *TT* is shown, e.g., by an argument of compactness, or in the following way.

If α is an ordinal > 0, the typed structure $\langle\langle V_\alpha, V_{\alpha+1}, V_{\alpha+2}, \ldots\rangle, \in\rangle$ is a model of *TT*. Since the class of ordinals cannot be embedded in the set of complete theories in $\mathcal{L}_{TT}$, there are ordinals, μ and ν, such that $\nu \leq \mu$, and such that the same sentences of $\mathcal{L}_{TT}$ are true in $\langle\langle V_\nu, V_{\nu+1}, \ldots\rangle, \in\rangle$ and $\langle\langle V_{\mu+1}, V_{\mu+2}, \ldots\rangle, \in\rangle$. Then, $V_\nu \subseteq V_\mu$. Thus we put $\mathcal{M} = \langle\langle V_\mu, V_{\mu+1}, \ldots\rangle, \in\rangle$ and $N = V_\nu$.

2 Interpreting NFU in SF From now on, we use the first-order language $\mathcal{L}_{NF}$.

Definitions $Eq(x) \equiv \forall y \forall z(y \sim z \rightarrow (y \in x \leftrightarrow z \in x))$,
$Set(x) \equiv \forall y(y \in x \rightarrow Eq(y))$,
$x \in_{Eq} y \equiv x \in y \wedge Set(y)$,
$x =_{Eq} y \equiv x \sim y$.

φ^{Eq} results from φ by replacing $\in$, $=$ by $\in_{Eq}$, $=_{Eq}$, and by restricting the quantifiers to $Eq(\ldots)$. Clearly, φ^{Eq} is stratifiable if φ is.

Theorem *If NFU $\vdash \varphi$, then SF $\vdash \varphi^{Eq}$, for every sentence φ.*

Proof: First, in order to settle the case of the logical theorems of *NFU*, we need $\exists x Eq(x)$ (which is obvious) and $\bigwedge_{1\leq i\leq n}(x_i \sim y_i \wedge Eq(x_i) \wedge Eq(y_i)) \rightarrow (\varphi[z_1 \ldots z_n := x_1 \ldots x_n]^{Eq} \leftrightarrow \varphi[z_1 \ldots z_n := y_1 \ldots y_n]^{Eq})$.

This is shown by an induction on φ, using the following theorems of *SF*:

$x \sim x' \wedge Eq(x) \rightarrow Eq(x')$,
$x \sim x' \wedge Set(x) \rightarrow Set(x')$,
$x \sim x' \wedge x \in_{Eq} y \wedge Eq(y) \rightarrow x' \in_{Eq} y$,
$y \sim y' \wedge x \in_{Eq} y \rightarrow x \in_{Eq} y'$.

Further, we show that the *Eq* versions of the comprehension axioms are true:

$$\forall z_1 \ldots \forall z_n\left(\bigwedge_{1\leq i\leq n} Eq(z_i) \rightarrow \exists y(Eq(y) \wedge \forall x(Eq(x) \rightarrow (x \in_{Eq} y \leftrightarrow \varphi^{Eq}))\right).$$

Suppose that $\bigwedge_{1\leq i\leq n} Eq(z_i)$ is true and let y be such that $\forall x(x \in y \leftrightarrow (\varphi^{Eq} \wedge Eq(x)))$. Therefore, we have $Eq(y)$, because:

$$\begin{aligned} x \in y \wedge x \sim x' &\rightarrow (\varphi^{Eq} \wedge Eq(x)) \\ &\rightarrow (\varphi^{Eq}[x := x'] \wedge Eq(x')), \\ &\rightarrow x' \in y. \end{aligned}$$

We also have $Set(y)$, hence $\forall x(Eq(x) \rightarrow (x \in_{Eq} y \leftrightarrow \varphi^{Eq}))$.

Finally, we have:

$$(\exists v\, v \in x \wedge x \sim y)^{Eq} \to Set(x) \wedge Set(y)$$

and

$$Set(x) \wedge Set(y) \wedge (x \sim y)^{Eq} \to x \sim y.$$

Therefore, the *Eq* version of the axiom of extensionality of *NFU* is true.

Permutations in *NFU* We work in *NFU*. p is a total function[1] iff p is a set of ordered pairs such that if $\langle x, y \rangle \in p$ and $\langle x, y' \rangle \in p$ then $y = y'$, and $\forall x \exists y \langle x, y \rangle \in p$.

We write $p(x) = y$ for $\langle x, y \rangle \in p$, $x \in rg(p)$ for $\exists y\, p(y) = x$ and $x \in p(y)$ for $\forall z(\langle y, z \rangle \in p \to x \in z)$.

Definition A total function p is a *quasi-permutation*[2] iff

$\forall x \forall y(p(x) \sim p(y) \to p(x) = p(y))$,
$\forall x \exists y\, p(y) \sim x$,
$\forall x \exists y\, p(p(y)) = p(x)$,
$\forall x \forall y(p(p(x)) = p(p(y))) \to p(x) = p(y)$,
$\exists y \forall x(x \notin rg(p) \to p(x) = y)$.

If $\forall x Ext(x)$ is true, then any quasi-permutation is simply a permutation in the ordinary sense. Therefore, we will readily suppose that the axiom of extensionality fails.

If p is a quasi-permutation, we let C_p denote the unique y such that $\forall x(x \notin rg(p) \to p(x) = y)$, and c_p the unique $x \in rg(p)$ such that $p(x) = C_p$. Intuitively speaking, c_p is the *concept* associated with p.

If p is a quasi-permutation, and if we replace in a formula φ, the atomic subformulas $x \in y$ and $x = y$ by $x \in p(y)$ and $x = y$, the result is a formula φ^p, which is stratifiable iff φ is.

Theorem *If p is a quasi-permutation, then $NFU \vdash \psi^p$, for every axiom of comprehension ψ, and $NFU \vdash (\forall x(\neg x \sim c_p \to Ext(x)))^p$.*

Proof: Let y' be such that $\forall x(x \in y' \leftrightarrow \varphi^p)$, and let y be such that $p(y) \sim y'$, then $\forall x(x \in_p y \leftrightarrow \varphi^p)$.

Next, we consider the axioms of weak extensionality. Let us assume that $\neg p(x) = C_p$ and $\neg p(y) = C_p$. Then, x and y belong to $rg(p)$. Therefore, if $p(x') = x$ and $p(y') = y$,

$$\begin{aligned} p(x) = p(y) &\to p(p(x')) = p(p(y')) \\ &\to p(x') = p(y') \\ &\to x = y. \end{aligned}$$

So, we have proved that $(\neg p(x) = C_p \wedge \neg p(y) = C_p \wedge p(x) = p(y)) \to x = y$.

From this we obtain: $\neg p(x) = C_p \to (p(x) = p(y) \to x = y)$, and $\forall x(\neg p(x) \sim p(c_p) \to \forall y(p(x) \sim p(y) \to x = y))$, which is $(\forall x(\neg x \sim c_p \to Ext(x)))^p$.

This theorem can be used to show the consistency of *SF* with various axioms of weak extensionality. We will give a few examples. For the sake of readability, we select an empty set: we let e be such that $\forall x\, x \notin e$.

Example 1 We define a quasi-permutation p in the following way: $p(x) = \{e\}$, if x is empty; $p(\{e\}) = e$; else $p(x) = x$. In this case, $C_p = \{e\}$, $c_p = e$ and $\forall x(x \in c_p \leftrightarrow x = c_p)^p$, which is written as $(c_p = \{c_p\})^p$. Therefore,

$$\exists z(z = \{z\} \wedge \forall x(\neg x \sim z \rightarrow Ext(x)))$$

is consistent with *SF*.

The theories $SF\varphi$ As a corollary of the theorem we have:

Proposition *Let φ be a stratifiable formula with at most the variable z occurring freely. Then, if $\forall z(z \in C_p \leftrightarrow \varphi^p)$, the following formula holds in NFU*:

$$\forall x(\neg \forall z(z \in x \leftrightarrow \varphi) \rightarrow Ext(x))^p.$$

If φ is stratifiable with no other variable than z free, we define $SF\varphi$ as $SF +$ the axiom $\forall x(\neg \forall z(z \in x \leftrightarrow \varphi) \rightarrow Ext(x))$. Thus, *NFU* is $NF_{z \neq z}$.

Clearly, $NF \vdash SF\varphi$, for every φ. The method of quasi-permutations enables one to show the consistency of $SF\varphi$, for various φ.

Example 2 V is the universal set: $\{x \mid x = x\}$. U is $\{z \mid \forall y\, y \notin z \wedge z \neq e\}$. Let $p(x) = V$ if $x \in U$, otherwise $p(x) = x$. In this case $C_p = V$ and all nonuniversal sets are extensional. Hence $SF + \forall x(\neg \forall v\, v \in x \rightarrow Ext(x))$ is consistent.

Example 3 The consistency of $SF\varphi$, φ being $\neg Ext(z)$. $p(x) = \{z \mid \forall v\, v \notin z\}$, if $\forall v\, v \notin x$; $p(\{z \mid \forall v\, v \notin z\}) = e$; else, $p(x) = x$. Then, $C_p = \{z \mid \forall v\, v \notin z\}$, and $z \in C_p \leftrightarrow \forall v\, v \notin z \leftrightarrow \exists y(p(z) = p(y) \wedge z \neq y) \leftrightarrow \neg Ext(z)^p$.

Example 4 The consistency of $SF\varphi$, φ being $Ext(z)$. $p(x) = \{z \mid \exists v\, v \in z\}$, if $\forall v\, v \notin z$; $p(\{z \mid \exists v\, v \in z\}) = e$; else, $p(x) = x$. Then, $C_p = \{z \mid \exists v\, v \in z\}$ and $z \in C_p \leftrightarrow \exists v\, v \in z \leftrightarrow \forall y(p(z) = p(y) \rightarrow z = y) \leftrightarrow Ext(z)^p$.

Example 5 The consistency of $SF\varphi$, φ being $\exists v\, v \in z$. $p(x) = \{z \mid z \neq e\}$, if x is in U; and otherwise, $p(x) = x$. We then have: $z \in C_p \leftrightarrow \exists v\, v \in p(z) \leftrightarrow (\exists v\, v \in z)^p$.

Example 6 If n is a natural number, let $\underline{n}(z)$ be the formula:

$$\exists y_1 \ldots \exists y_n \left(\bigwedge_{0 \leq i < j \leq n} y_i \neq y_j \wedge \forall v \left(v \in z \leftrightarrow \bigvee_{1 \leq i \leq n} v = y_i \right) \right)$$

The numeral $\underline{n}$ is $\{z \mid \underline{n}(z)\}$. We define p as follows: $p(x) = \underline{n}$, if $x \in U$; otherwise, $p(x) = x$. Then, $(z \in \underline{n})^p \leftrightarrow z \in \underline{n} \leftrightarrow z \in C_p$. This shows that $SF\varphi$ is consistent if φ is the formula $z \in \underline{n}$.

Theorem *There is a φ such that $SF\varphi = NF$.*

Proof: We let φ be the stratifiable formula

$$\forall x(\neg Ext(x) \rightarrow z \notin x),$$

and we derive $\forall x Ext(x)$ in $SF\varphi$.

The axiom of weak extensionality of $SF\varphi$ implies

$$\forall x(\neg Ext(x) \to \forall z(z \in x \leftrightarrow \forall x(\neg Ext(x) \to z \notin x))).$$

From this we have in the first place:

$$\forall x(\neg Ext(x) \to (z \in x \to (\neg Ext(x) \to z \notin x))), \text{ and } \forall x(\neg Ext(x) \to z \notin x).$$

On the other hand: $\forall x(\neg Ext(x) \to (\forall x(\neg Ext(x) \to z \notin x) \to z \in x))$, and $\forall x(\neg Ext(x) \to z \in x)$. Therefore, $\forall x Ext(x)$.

NOTES

1. There is no problem in dealing with the notions of an ordered pair and a total function in *NFU*, since these objects are not empty, at least if the ordered pair $\langle x, y\rangle$ is defined as the set $\{\{x\},\{x,y\}\}$.
2. The requirements that follow will ensure that the range $rg(p)$ of p is the set of nonempty objects plus a fixed empty object, which can be thought of as *the* empty set. This is stated in the first two conditions. The next two conditions say that p is a permutation on $rg(p)$. The last condition states that the atoms (i.e., the empty objects other than the empty set) are all sent to the same set. This definition is quite general in the sense that it works not only for *NFU*, but also for *SF* with the axiom $\exists z \forall x(\neg x \sim z \to Ext(x))$.

REFERENCES

[1] Boffa, M., "The consistency problem for *NF*," *The Journal of Symbolic Logic*, vol. 42 (1977), pp. 215–220.

[2] Boffa, M., "*ZFJ* and the consistency problem for *NF*," *Jahrbuch der Kurt Gödel Gesellschaft* (1988), pp. 102–106.

[3] Crabbé, M., "Typical ambiguity and the Axiom of Choice," *The Journal of Symbolic Logic*, vol. 49 (1984), pp. 1074–1078.

[4] Jensen, R.B., "On the consistency of a slight (?) modification of *NF*," *Synthese*, vol. 19 (1968–69), pp. 250–263.

[5] Quine, W.V.O., *From a Logical Point of View*, Harvard University Press, Cambridge, 1953.

[6] Specker, E., "The axiom of choice in Quine's 'New foundations for mathematical logic'", *Proceedings of the National Academy of Science of the United States of America*, vol. 39 (1953), pp. 972–975.

[7] Specker, E., "Typical ambiguity," pp. 116–124 in *Logic, Methodology and the Philosophy of Science*, edited by E. Nagel, P. Suppes, and A. Tarski, Stanford University Press, Stanford, 1962.

Université Catholique de Louvain
Département de Philosophie
Centre de Logique
Chemin d'Aristote, 1
B-1348 Louvain-la-Neuve, Belgium