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STRATIFICATION AND CUT-ELIMINATION

MARCEL CRABBÉ

Introduction. In this paper, we show the normalization of proofs of NF (Quine's *New Foundations*; see [15]) minus extensionality. This system, called SF (*Stratified Foundations*) differs in many respects from the associated system of simple type theory.[1] It is written in a first order language and not in a multi-sorted one, and the formulas need not be stratifiable, except in the instances of the comprehension scheme. There is a universal set, but, for a similar reason as in type theory, the paradoxical sets cannot be formed.

It is not immediately apparent, however, that SF is essentially richer than type theory. But it follows from Specker's celebrated result (see [16] and [4]) that the stratifiable formula (*extensionality* $\rightarrow$ *the universe is not well-orderable*) is a theorem of SF.

It is known (see [11]) that this set theory is consistent, though the consistency of NF is still an open problem.[2]

The connections between consistency and cut-elimination are rather loose. Cut-elimination generally implies consistency. But the converse is not true. In the case of set theory, for example, ZF-like systems, though consistent, cannot be freed of cuts because the separation axioms allow the formation of sets from unstratifiable formulas. There are nevertheless interesting partial results obtained when restrictions are imposed on the removable cuts (see [1] and [9]). The systems with stratifiable comprehension are the only known set-theoretic systems that enjoy full cut-elimination.

Since cut-elimination for stratified set theory trivially implies cut-elimination for type theory, one justly expects that we will extend Girard's method (see [7], [8], [12], [14], and [19]), which mirrors in proof theory the construction, by iteration of the power set operation, of a natural model of type theory. The extension we give

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[1]The types of this associated system are the natural numbers, $\in$ is its sole relation, and the only nonlogical axioms are the axioms of comprehension.

[2]Actually Jensen's result (in [11]) seems to show more. The system proved consistent is NFU, i.e. NF with *Urelements*. In this system, the only nonextensional objects are the atoms, i.e. the empty sets.

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exploits proof-theoretically the technique used by Jensen (in [11]) for the construction of ω-models of SF.[3]

Girard's proof of the cut-elimination theorem for type theory requires the existence of an ω-model of type theory. This seems unavoidable, since the normalization of type theory is much stronger than the consistency of type theory,[4] which is elementarily provable. The normalization of intuitionistic type theory—without extensionality—elementarily implies the consistency of classical extensional higher order arithmetic.[5] In the case of NF, the situation is a little bit different. The normalization of the nonextensional intuitionistic system implies of course the consistency of the system; and, because Gödel's negative interpretation works in this case, this in turn implies the consistency of the classical nonextensional system. However, contrary to what happens in type theory, it is not known how to extensionalize models of the nonextensional fragment of NF without destroying comprehension axioms.

The proof that we present here below does not require more than the existence of an ω-model of NFU (see footnote 2). In fact, the result can be established in NFU plus Rosser's axiom. But since we do not suppose that the reader is acquainted with the NF literature, we will not follow this path here. We will instead carry out the proof directly in ZF.

§1. Naive set theory. As everybody knows, naive set theory is inconsistent. Nevertheless, we will recall the language and the natural deduction rules for this system essentially because its proof theory is not trivial and because every set theory worth studying must seemingly be a part of a system of this kind. The material of this section is therefore usable for the study of consistent fragments.

1.1. *Terms and formulas.* Well-formed expressions are built up from a denumerably infinite sequence of variables and the symbols $\rightarrow$, $\forall$, $\in$, $\{\ |\ \}$, (, and).

The notions of term and formula are defined as follows:

A variable is a term.

If P and Q are terms, then $P \in Q$ is a formula.

If A and B are formulas and if x is a variable, then $(A \rightarrow B)$ and $\forall x A$ are formulas (the occurrences of x in $\forall x A$ are bound).

If A is a formula and x a variable, then $\{x \mid A\}$ is a term (called abstract; the occurrences of x in $\{x \mid A\}$ are bound).

[3] Jensen's idea is, following Specker's suggestion (in [17]), to construct a model of type theory with a shifting automorphism sending objects of a given type to the next one (the types are the natural numbers). He achieves this by showing how to make the types indiscernible—thereby loosing extensionality. The main tool used for that is Ramsey's theorem. This is sufficient for the proof of the consistency of the system. The method generalizes, however, and enables one to construct an ω-model as well. But in this case Ramsey's theorem is not strong enough, and an exploitation of the Erdös-Rado theorem is necessary.

[4] See footnote 1.

[5] This can be seen by first adding the axioms of arithmetic without induction, then performing a negative interpretation of the classical system (which works pretty well when extensionality is not present), and, finally, making an extensional interpretation.

A *sentence* [*closed term*] is a formula [term] having no free occurrences of variables.

1.2. *Derivations and cut elimination.* We now proceed to the formulation of the natural deduction rules for naive set theory. These rules are according to the gist of natural deduction not relations between formulas but procedures to construct derivations, i.e. formal proofs. Derivations will be coded in an extended typed λ-calculus along the line of an idea of Curry (see [5, 9E]) revisited by Howard in [10]. We now present the language for the formulation of the derivations.

For each formula A, there is a denumerably infinite sequence of assumptions, called *assumptions of* A. To be formal about it we identify the ith assumption of A with the ordered pair (i, A). The same letters will be used for assumptions and variables (we hope that no confusion will arise). Though we will use the ordinary mode of speech concerning the bound and free occurrences, we will tacitly suppose that this problem is solved by using a method like that of Bourbaki in [2, Chapter 1, §1] or de Bruijn in [6]—the net result being that expressions differing only with regard to bound variables are identified.

We define the notions of *derivation* of a formula (its *conclusion*) and of *free* and *bound* occurrences of an assumption or of a variable in a derivation simultaneously:

An *assumption* x of A is a derivation of A in which x is the sole (free) occurrence of an assumption; the free occurrences of variables in x are those in A.

Introduction of $\rightarrow$. If Σ is a derivation of conclusion B and x is an assumption of A, then $\lambda x\Sigma$ is a derivation of $(A \rightarrow B)$; the occurrences of x in $\lambda x\Sigma$ are bound, and the other ones (of assumptions or of variables) remain bound or free as they are in A and Σ.

Elimination of $\rightarrow$. If Σ is a derivation of the conclusion $(A \rightarrow B)$ and Π a derivation of A, then $(\Sigma\Pi)$ is a derivation of B; an occurrence is free or bound in $(\Sigma\Pi)$ iff it is so in Σ or Π.

Introduction of $\forall$. If Σ is a derivation of the conclusion A and x is a variable not occurring free in a free occurrence of an assumption in Σ, then $\forall x\Sigma$ is a derivation of $\forall xA$; the occurrences of x are bound in $\forall x\Sigma$, the other ones (of assumptions or of variables) remain bound or free as they are in Σ.

Elimination of $\forall$. If Σ is a derivation of the conclusion $\forall xA$ and P is a term, then (ΣP) is a derivation of the conclusion $A[x := P]$; the occurrences remain free or bound as they are in Σ or P. ($A[x := P]$ is the result of the substitution of P for x at the free occurrences of x in A, up to a renaming of bound variables if necessary.)

Introduction of $\{\,|\,\}$. If Σ is a derivation of $A[x := P]$, then $(\uparrow P \in \{x \mid A\})\Sigma$ is a derivation of $P \in \{x \mid A\}$; the free and bound occurrences of assumptions in $(\uparrow P \in \{x \mid A\})\Sigma$ are the same as in Σ; the free and bound occurrences of variables in $(\uparrow P \in \{x \mid A\})\Sigma$ are the same as in Σ or $P \in \{x \mid A\}$. We will abbreviate $(\uparrow P \in \{x \mid A\})\Sigma$ as $\uparrow\Sigma$

Elimination of $\{\,|\,\}$. If Σ is a derivation of the conclusion $P \in \{x \mid A\}$, then $(\Sigma\downarrow)$ is a derivation of $A[x := P]$; free and bound occurrences in $(\Sigma\downarrow)$ are the same as in Σ.

A *closed* derivation is a derivation without free (occurrences of) assumptions. A is a *theorem* of (a fragment of) naive set theory if there is a closed derivation of A.

For the sake of readability, we will use the current conventions on parentheses: omitting them when they are necessary, adding them when not necessary;

particularly, when parentheses are missing in expressions of the form $T_1 T_2 \cdots T_n$ we associate to the left: $(\cdots(T_1 T_2)\cdots T_n)$.

If Σ and Π are derivations, and x is an assumption of the same formula as the conclusion of Π, then $\Sigma[x := \Pi]$ is the derivation resulting from the substitution in Σ of Π at the free occurrences of x—possibly up to a renaming of bound assumptions.

If Σ is a derivation, P is a term and x is a variable, then $\Sigma[x := P]$ is the derivation resulting from the substitution in Σ of P at the free occurrences of x. Formally $\Sigma[x := P]$ is defined by induction on Σ, starting from the initial clause: $(i, A)[x := P] = (i, A[x := P])$.

1.3. *Cut elimination.* We are now in a position to formulate the

CUT-ELIMINATION RULES. a) $(\lambda x\Sigma)\Pi$ *immediately reduces to* $\Sigma[x := \Pi]$.

b) $(\forall x\Sigma)P$ *immediately reduces to* $\Sigma[x := P]$.

c) $(\uparrow\Sigma)\downarrow$ *immediately reduces to* Σ.

A *cut* in a derivation is an occurrence in the derivation of a derivation of one of the forms indicated at the left in the cut-elimination rules, i.e. $(\lambda x\Sigma)\Pi$, $(\forall x\Sigma)P$ or $(\uparrow\Sigma)\downarrow$. A derivation is *normal* if it contains no cuts. A derivation Σ *reduces in one step to* Π iff Π is obtained by removing a cut in Σ as prescribed by the cut-elimination rules. A *reduction sequence* is a sequence of derivations $\Sigma_0, \Sigma_1, \ldots$ such that Σ_i reduces in one step to Σ_{i+1}, if Σ_{i+1} is in the sequence. Σ *reduces to* Π if there is a reduction sequence starting with Σ and ending in Π. A derivation Σ is *strongly normalizable* iff each reduction sequence starting with Σ is finite.

Clearly, a strongly normalizable derivation reduces to a normal one.

REMARK. It is well known that the inconsistency of naive set theory is a consequence of Russell's paradox. In the present context, it is formulated as follows. Let A be any formula and consider the abstract R_A: $\{z \mid (z \in z \to A)\}$. Let x be an assumption of $R_A \in R_A$; then $\lambda x(x\downarrow x)(\uparrow\lambda x(x\downarrow x))$ is a closed derivation of A. However, this derivation does not reduce to a normal one. More generally, it is elementarily provable that there is no normal closed derivation of $\forall x\forall y x \in y$ (one cannot cut-freely prove an absurdity). Therefore the collection of cut-free provable theorems of naive set theory is a natural paraconsistent set theory (see [13, pp. 94–95]).

§2. Stratified set theory. Roughly speaking, a stratifiable formula is a formula that one gets when one erases the types in a formula of type theory.

DEFINITIONS. A *weak stratification* assignment for a formula C is a function from the occurrences of terms in the formula to the integers satisfying the following requirements:

At an occurrence of the formula $P \in Q$ in C, the value of Q is i iff the value of P is $i - 1$.

At an occurrence of the term $\{x \mid A\}$ in C, each occurrence of x in $\{x \mid A\}$ has the same value i and the value of $\{x \mid A\}$ is $i + 1$.

At an occurrence of the formula $\forall xA$ in C, each occurrence of x has the same value.

A *stratification* assignment for C is a weak stratification assignment such that, for every variable x, all occurrences of x in C have the same value.

Similarly, a *weak stratification* [*stratification*] assignment for an abstract $\{x \mid A\}$ is a weak stratification [stratification] assignment for A such that each occurrence of x has the same value.

A formula or term is *weakly stratifiable* [*stratifiable*] if there is a weak stratification [stratification] assignment for it.

For example, Russell's terms $\{z \mid (z \in z \to A)\}$ are not weakly stratifiable, but the formula $(z \in z \to A)$ is weakly stratifiable—though not stratifiable—if A is weakly stratifiable.

Clearly, a sentence or closed term is stratifiable iff it is weakly stratifiable.

The system SF. SF (stratified foundations) is the fragment of naive set theory that results from the restriction that the abstracts have to be weakly stratifiable.

REMARKS. 1. It is quite routine to show that the theorems of SF are exactly the formulas provable in nonextensional intuitionistic NF (with abstracts) without $\neg$, $\wedge$, $\vee$ and $\exists$. We could have added more logical constants and/or the classical double negation rule and appended the appropriate cut-elimination rules. Everything would then have worked out rather well, but the matter would have become much more involved without adding anything new to what happens in other systems (see [9] for a detailed presentation).

2. In order that the derivations behave nicely through reduction, which supposes that the abstracts are closed under substitution, we have formulated the system with the concept of weak stratification instead of that of stratification. If we had used only stratifiable terms, we would not have lost any theorem, but then $\Sigma[x := P]$ would not necessarily be a derivation if $(\forall x \Sigma)P$ is a derivation, and some theorems would not be cut-free provable for the trivial reason that the cut-elimination rules would be undefined for some cuts (more on this point in [3]).

§3. Main theorem. This section is devoted to the proof of a theorem which may be seen as an abstract reformulation of the Gentzen-Prawitz Hauptsatz in the line of Tait [18] and Girard [7].

DEFINITION. A *sorted structure* $\mathcal{N}$ is a set N with a function *sort* sending the elements of N *onto* the terms of SF, and with, for each derivation Σ, a relation $\in_\Sigma$ on N such that, if $\beta \in_\Sigma \alpha$, then Σ is a derivation of $\mathrm{sort}(\beta) \in \mathrm{sort}(\alpha)$.[6]

We call P the *sort* of α if $\mathrm{sort}(\alpha) = P$.

DEFINITION. A *valuation* is a function whose domain is a finite set of variables. If v and v' are valuations agreeing on all the variables (other than x) for which both are defined and if v' is defined for x, then we write $v \prec_x v'$.

A valuation is *defined* for a formula [term] iff it is defined for every variable occurring free in the formula [term]. □

If $\mathcal{N}$ is a sorted structure and v is a valuation defined for A and taking values in N, we will denote by $A[v]$ the formula resulting from A through the simultaneous substitution for each free variable x in A of the term $\mathrm{sort}(v(x))$. $P[v]$ is defined analogously.

[6] $\beta \in_\Sigma \alpha$ intuitively means that β belongs to α for the reason Σ.

DEFINITION. A *normalization structure* is a sorted structure $\mathcal{N}$ equipped with an operator $[\![\,,\,]\!]$, such that for each term P and valuation v defined for P and taking values in N, $[\![P, v]\!]$ belongs to N and is of sort $P[v]$. □

Let $\mathcal{N}$ be a normalization structure. If v is a valuation into N defined for C and if Σ is a derivation of $C[v]$, we define inductively the relation $\Sigma, v \Vdash C$ as follows:[7]

$\Sigma, v \Vdash P \in Q$ iff $[\![P, v]\!] \in_\Sigma [\![Q, v]\!]$.

$\Sigma, v \Vdash (A \to B)$ iff, for every Π such that $\Pi, v \Vdash A$, $(\Sigma\Pi), v \Vdash B$.

$\Sigma, v \Vdash \forall x A$ iff for every valuation v' such that $v'(x)$ belongs to N and $v \prec_x v'$, $\Sigma\ \mathrm{sort}(v'(x)), v' \Vdash A$.

In order to be able to state the theorem below, we have to introduce some further definitions concerning the normalization structures.

3.1. *Comprehensiveness.* The normalization structure $\mathcal{N}$ is *comprehensive* iff (a) the comprehension axiom holds in the sense that

$$\alpha \in_\Sigma [\![\{x \mid A\}, v]\!] \quad \text{iff} \quad \Sigma\downarrow, v' \Vdash A \qquad (\text{for } v \prec_x v' \text{ and } v'(x) = \alpha),$$

and (b) $[\![P[x := Q], v]\!] = [\![P, v']\!]$, if $v \prec_x v'$ and $v'(x) = [\![Q, v]\!]$.

SUBSTITUTION LEMMA. *If $\mathcal{N}$ is comprehensive, then*:

1. $\Sigma, v \Vdash A[x := Q]$ *iff* $\Sigma, v' \Vdash A$ (*if* $v \prec_x v'$ *and* $v'(x) = [\![Q, v]\!]$);
2. *if* $\Sigma, v \Vdash \forall x A$ *and* v *is defined for* Q, *then* $\Sigma(Q[v]), v \Vdash A[x := Q]$;
3. $\Sigma, v \Vdash Q \in \{x \mid A\}$ *iff* $\Sigma\downarrow, v \Vdash A[x := Q]$.

The inductive proof of 1 is straightforward. The two other parts follow from the first.

3.2. *Stability.* We introduce first the notion of *critical reduction*.

(a) $(\lambda x\Sigma)\Pi T_1 \cdots T_n$ *critically reduces* to $\Sigma[x := \Pi]T_1 \cdots T_n$, provided Π is a strongly normalizable derivation.[8]

(b) $(\forall x\Sigma)PT_1 \cdots T_n$ *critically reduces* to $\Sigma[x := P]T_1 \cdots T_n$.

(c) $(\uparrow\Sigma)\downarrow T_1 \cdots T_n$ *critically reduces* to $\Sigma T_1 \cdots T_n$.

In the possibly empty sequence $T_1 \cdots T_n$, the T_i's are either derivations, or terms, or the symbol $\downarrow$.

DEFINITION. The normalization structure $\mathcal{N}$ is *stable* iff the following condition holds: if $\alpha \in_\Pi \beta$ and Σ critically reduces to Π, then $\alpha \in_\Sigma \beta$.

STABILITY LEMMA. *If $\mathcal{N}$ is stable, then if $\Pi, v \Vdash C$ and Σ critically reduces to Π, then $\Sigma, v \Vdash C$.*

PROOF (Induction on C). The starting case, when C is $P \in Q$, is nothing more than a reformulation of the hypothesis.

Suppose that $\Pi, v \Vdash (A \to B)$ and that Σ critically reduces to Π. If $\Theta, v \Vdash A$, then $\Pi\Theta, v \Vdash B$. By the induction hypothesis, $\Sigma\Theta, v \Vdash B$ because $\Sigma\Theta$ critically reduces to $\Pi\Theta$.

Suppose that Σ critically reduces to Π, and that $\Pi, v \Vdash \forall x A$. If $v \prec_x v'$, then $\Sigma\mathrm{sort}(v'(x))$ critically reduces to $\Pi\mathrm{sort}(v'(x))$ and $\Pi\mathrm{sort}(v'(x)), v' \Vdash A$. By the induction hypothesis $\Sigma\mathrm{sort}(v'(x)), v' \Vdash A$.

[7] $\Sigma, v \Vdash C$ may be read: "Σ forces C, relative to v" or "Σ is a valid proof of C, modulo v".

[8] The necessity of this restriction will become apparent in §§4 and 5 (see, for example, footnote 10).

3.3. *Soundness.*

DEFINITION. An *analytic* derivation is a derivation of the form $xT_1 \cdots T_n$, where x is an assumption and the T_i's are either derivations, terms or the symbol $\downarrow$. A *strongly analytic derivation* is a strongly normalizable analytic derivation. □

In particular, every assumption is strongly analytic.

DEFINITION. The normalization structure $\mathcal{N}$ is *sound* iff the following two conditions hold:

(a) If $\alpha \in_\Sigma \beta$, then Σ is strongly normalizable.

(b) If Σ is strongly analytic, then $\alpha \in_\Sigma \beta$.

SOUNDNESS LEMMA. *If $\mathcal{N}$ is sound, then*

1. *if $\Sigma, v \Vdash C$, then Σ is strongly normalizable, and*
2. *if Σ is a strongly analytic derivation of $C[v]$, then $\Sigma, v \Vdash C$.*

PROOF. The two parts are proved conjointly by induction on C. Suppose that Σ, $v \Vdash (A \to B)$. By the second induction hypothesis, Σx, $v \Vdash B$ if x is an assumption of sort $A[v]$. It follows by the first induction hypothesis that Σx is strongly normalizable, whence so is Σ.

Let Σ be a strongly analytic derivation of $(A \to B)[v]$. Then, by the first induction hypothesis, if Π, $v \Vdash A$, then Π is a strongly normalizable derivation of $A[v]$. It follows that $\Sigma\Pi$ is strongly analytic. Therefore, by the second induction hypothesis, $\Sigma\Pi$, $v \Vdash B$.

If Σ, $v \Vdash \forall xA$, then $\Sigma\mathrm{sort}(v'(x))$, $v' \Vdash A$ (if $v \prec_x v'$ and $v'(x)$ in N). By the first induction hypothesis, $\Sigma\mathrm{sort}(v'(x))$ is strongly normalizable, and so is Σ.

Let Σ be a strongly analytic derivation of $\forall xA[v]$. Then, for every term P, ΣP is strongly analytic too. Hence, by the second induction hypothesis, ΣP, $v' \Vdash A$ for every v' such that $v \prec_x v'$ and $v'(x)$ is of sort P. □

DEFINITION. A *normalization model* is a comprehensive, stable and sound normalization structure. □

Let $\mathcal{N}$ be a normalization structure. If v is a valuation into N, $\Sigma[v]$ will denote the derivation resulting from the simultaneous replacement of each free variable x of Σ, for which v is defined, by $\mathrm{sort}(v(x))$.

If s is a function such that for every assumption x (for which s is defined), $s(x)$ is a derivation of the same conclusion as x, then $\Sigma[s]$ is the derivation resulting from the simultaneous substitution of $s(x)$ for each free assumption x in Σ for which s is defined.

THEOREM. *If $\mathcal{N}$ is a normalization model and Σ is a derivation of C, then $\Sigma[v][s]$, $v \Vdash C$ for each v (defined for C) and s such that $s(x[v])$, $v \Vdash A$ (if x is an assumption of A and s is defined for the assumption $x[v]$).*

PROOF (Induction on the construction of Σ). If Σ is an assumption, $\Sigma[v][s]$, $v \Vdash C$ by hypothesis or the soundness lemma, according to whether s is defined or not for $\Sigma[v]$.

Introduction of $\to$. Σ is $\lambda x\Pi$ and C is $(A \to B)$. Let Θ, $v \Vdash A$. We want to prove that $(\lambda x\Pi)[v][s]\Theta$, $v \Vdash B$. By the soundness lemma, Θ is strongly normalizable. It is therefore sufficient, due to the stability lemma, to show that $\Pi[v][s']$, $v \Vdash B$, where s' is like s with the possible exception that $s'(x[v]) = \Theta$. The result follows from the induction hypothesis.

Elimination of $\to$. Σ *is* $\Pi\Theta$, Π is of $(A \to C)$ and Θ is of A. By the induction hypothesis, $\Pi[v][s], v \Vdash (A \to C)$ and $\Theta[v][s], v \Vdash A$, whence $\Sigma[v][s], v \Vdash C$.

Introduction of $\forall$. Σ *is* $\forall x\Pi$ and C is $\forall xA$. We have to show that $(\forall x\Pi)[v][s]\mathrm{sort}(v'(x)), v' \Vdash A$, if $v \prec_x v'$ and $v'(x)$ belongs to N. But as x is not free in a free assumption of Π, $(\forall x\Pi)[v][s]\mathrm{sort}(v'(x))$ critically reduces to $\Pi[v'][s]$. The result then follows from the stability lemma and the induction hypothesis.

Elimination of $\forall$. Σ is ΠP, Π is of $\forall xA$ and C is $A[x := P]$. By the induction hypothesis, $\Pi[v][s], v \Vdash \forall xA$, whence $\Sigma[v][s], v \Vdash A[x := P]$, by the substitution lemma.

Introduction of $\{\ |\ \}$. Σ is $(\uparrow P \in \{x \mid A\})\Pi$ and C is $P \in \{x \mid A\}$. We wish to show that $\uparrow\Pi[v][s], v \Vdash P \in \{x \mid A\}$. By the substitution lemma, it suffices to show that $\uparrow\Pi[v][s]\downarrow, v \Vdash A[x := P]$. By the stability lemma it is sufficient to have that $\Pi[v][s], v \Vdash A[x := P]$, which is the induction hypothesis.

Elimination of $\{\ |\ \}$. Σ is $\Pi\downarrow$, Π of $P \in \{x \mid A\}$ and C is $A[x := P]$. It suffices here to apply the substitution lemma to the induction hypothesis.

COROLLARY. *If there is a normalization model, then every derivation is strongly normalizable.*

PROOF. Let Σ be a derivation of C. We first apply the theorem to Σ with a v defined for C such that $\mathrm{sort}(v(x)) = x$ and s empty. We conclude with the soundness lemma. $\square$

The rest of the paper is devoted to the proof of the hypothesis of the corollary.

§4. The admissible sets. We fix a set theory which we call $\mathfrak{T}$ and which is to be sufficiently strong for the formalization of the definitions of this section. ZF will do quite well but is unnecessarily strong. All that is needed is extensionality, pair formation, powerset, union, Δ_0-separation and a convenient axiom of infinity (existence of ω). Fix a model $\mathfrak{M}$ of $\mathfrak{T}$ in which ω is standard, i.e. the object which codes in $\mathfrak{M}$ the set of finite ordinals of $\mathfrak{M}$ is isomorphic to ω. In this section we will work inside $\mathfrak{M}$. Particularly, we will tacitly suppose that sets, relations and functions are to be elements of $\mathfrak{M}$. Syntactic objects like terms, formulas and derivations are identified in a canonical way with finite ordinals or, more naturally, with hereditarily finite sets (of $\mathfrak{M}$). For the time being, we suppose moreover than $\mathfrak{M}$ has elements u_i (for each integer i) such that (in $\mathfrak{M}$) $\mathscr{P}\mathscr{P}\mathscr{P}(u_i) \subseteq u_{i+1}$. Hence $\omega \subseteq u_i$.[9]

DEFINITIONS. 1. A set α is *of sort P over u_i* iff α is a set of ordered pairs (Σ, β) such that, for some Q, Σ is a derivation of the formula $Q \in P$ and β is an element of u_i. α is *sorted over u_i* iff there is a term P such that α is of sort P over u_i.

We will write $\mathrm{sort}(\alpha) = P$ if α is of sort P over u_i. $\mathrm{Sor}(i)$ denotes the set of sorted sets over u_i. Clearly, $\mathrm{Sor}(i)$ is not empty and, with Kuratowski's ordered pair, $\mathrm{Sor}(i) \subseteq \mathscr{P}\mathscr{P}\mathscr{P}(u_i) \subseteq u_{i+1}$.

2. A set α is *quasi-admissible* over u_i iff α belongs to $\mathrm{Sor}(i)$ and if for every (Σ, β) in α, β belongs to $\mathrm{Sor}(i-1)$ and Σ is a derivation of the formula $\mathrm{sort}(\beta) \in \mathrm{sort}(\alpha)$.

3. A quasi-admissible set α is *stable* iff for every Σ and β, (Σ, β) belongs to α whenever there is a derivation Π such that (Π, β) belongs to α and Σ critically reduces to Π.

[9] The proof that such an $\mathfrak{M}$ exists is postponed to the Appendix.

4. A quasi-admissible set (over u_i) α is *sound* iff the following two conditions hold:

(a) For every Σ and β, Σ is strongly normalizable whenever (Σ, β) belongs to α.

(b) For every β in $\mathrm{Sor}(i-1)$ and every strongly analytic derivation Σ of $\mathrm{sort}(\beta) \in \mathrm{sort}(\alpha)$, (Σ, β) belongs to α.

5. An *admissible set* over u_i is a stable sound quasi-admissible set over u_i. □

$\mathrm{Adm}(i)$ is the set of admissible sets over u_i. Trivially, $\mathrm{Adm}(i) \subseteq \mathrm{Sor}(i) \subseteq u_{i+1}$. Using the crucial fact that a derivation which critically reduces to a strongly normalizable one is strongly normalizable too, one proves that for every term P there is an admissible set of sort P over u_i.[10]

For every integer i and derivation Σ one defines the relation $\in_{i,\Sigma}$ between elements of $\mathrm{Adm}(i)$ and $\mathrm{Adm}(i+1)$ as follows:

$$\alpha \in_{i,\Sigma} \beta \quad \text{iff} \quad (\Sigma, \alpha) \text{ belongs to } \beta.$$

Let σ be a stratification assignment for a (*stratifiable*) formula C [term P] and v be a valuation defined for C [for P] such that $v(x)$ belongs to $\mathrm{Adm}(\sigma(x))$, for every variable x occurring free in C [in P]. Every valuation of this kind is in $\mathfrak{M}$. $A[v]$ and $P[v]$ are defined in a similar way as in §3, and we will define now $\Sigma, \sigma, v \Vdash C$ and $[\![P, \sigma, v]\!]$ by induction on the length of C and P (Σ is a derivation of $C[v]$). Since we wish that $[\![P, \sigma, v]\!]$ be an admissible set of sort $P[v]$, we will have to make sure that:

1. If $\Pi, \sigma, v \Vdash C$ and Σ critically reduces to Π, then $\Sigma, \sigma, v \Vdash C$;
2. If $\Sigma, \sigma, v \Vdash C$, then Σ is strongly normalizable;
3. If Σ is a strongly analytic derivation of $C(v)$, then $\Sigma, \sigma, v \Vdash C$; and
4. $[\![P, \sigma, v]\!]$ is of sort $P[v]$ and belongs to $\mathrm{Adm}(\sigma(P))$.

These facts are to be established while we give the definition. However, when C is of the form $P \in Q$, $(A \to B)$ or $\forall x A$, the proofs are trivial or similar to the corresponding cases of the stability and soundness lemmas in §3. We shall accordingly omit these cases.

$[\![x, \sigma, v]\!] = v(x)$.

$\Sigma, \sigma, v \Vdash P \in Q$ iff $[\![P, \sigma, v]\!] \in_{\sigma(P), \Sigma} [\![Q, \sigma, v]\!]$.

$\Sigma, \sigma, v \Vdash (A \to B)$ iff, for every Π such that $\Pi, \sigma, v \Vdash A$, we have $(\Sigma\Pi), \sigma, v \Vdash B$.

$\Sigma, \sigma, v \Vdash \forall x A$ iff, for every valuation v' such that $v \prec_x v'$ and $v'(x)$ belongs to $\mathrm{Adm}(\sigma(x))$, we have

$$\Sigma \mathrm{sort}(v'(x)), \sigma, v' \Vdash A.$$

$[\![\{x \mid A\}, \sigma, v]\!]$ is the set of those ordered pairs (Σ, α) such that α belongs to $\mathrm{Adm}(\sigma(x))$, Σ is of $\mathrm{sort}(\alpha) \in \{x \mid A[v]\}$ and $\Sigma\!\downarrow, \sigma, v' \Vdash A$, if $v \prec_x v'$ and $v'(x) = \alpha$.

We have to show that $[\![\{x \mid A\}, \sigma, v]\!]$ is admissible. First, if Π critically reduces to Σ, then it is also the case that $\Pi\!\downarrow$ critically reduces to $\Sigma\!\downarrow$ and the conclusion that $[\![\{x \mid A\}, \sigma, v]\!]$ is stable follows from the induction hypothesis.

Next, we notice that if (Σ, α) belongs to $[\![\{x \mid A\}, \sigma, v]\!]$, then, by the induction hypothesis, $\Sigma\!\downarrow$ is strongly normalizable and therefore so is Σ.

[10] For example, the set of the ordered pairs (Σ, β) such that β belongs to $\mathrm{Sor}(i-1)$ and Σ is a strongly normalizable derivation of the formula $\mathrm{sort}(\beta) \in P$.

Finally, if Σ is strongly analytic, then $\Sigma\downarrow$ is also strongly analytic and $\Sigma\downarrow$, σ, $v' \Vdash A$ by the induction hypothesis.

§5. Main definition. In this section, we will define explicitly a normalization model $\mathcal{N}$. Though we will use the results obtained inside $\mathfrak{M}$ in §4, we will work here in what is called the real world.

The properties of $\mathfrak{M}$ used so far suffice largely for the proof of the normalization of type theory. In the case of SF, we will suppose further that there is an automorphism π of $\mathfrak{M}$ such that, for every integer i, $\pi(u_i) = u_{i+1}$. Clearly, the function π cannot be in $\mathfrak{M}$. The proof that such a model $\mathfrak{M}$ exists (which is not quite obvious) can be obtained by using a method for constructing ω-models with indiscernibles as is done in [11]. For the sake of completeness, we sketch a proof in the Appendix.

Since $\mathfrak{M}$ is ω-standard, the proof-theoretical entities are fixed by π, i.e. $\pi(A) = A$, $\pi(P) = P$ and $\pi(\Sigma) = \Sigma$, for a formula A, a term P and a derivation Σ. Moreover $\pi(\mathrm{Sor}(i)) = \mathrm{Sor}(i+1)$, $\pi(\mathrm{Adm}(i)) = \mathrm{Adm}(i+1)$, and $\alpha \in_{i,\Sigma} \beta$ iff $\pi(\alpha) \in_{i+1,\Sigma} \pi(\beta)$.

DEFINITION. N is Adm(0), and sort is the restriction to N of the function sort defined in §4.[11]

$\alpha \in_\Sigma \beta$ iff $\alpha \in_{0,\Sigma} \pi(\beta)$, i.e. iff (Σ, α) belongs to $\pi(\beta)$, for α and β in N. □

Clearly, sort is onto (see footnote 10). The relations $\in_\Sigma$ are not coded in $\mathfrak{M}$, and we are not allowed to give the definition of $[\![P, v]\!]$ below for arbitrary terms of naive set theory. However it will work for the weakly stratifiable ones.

With every term P of SF we associate a *stratifiable* term P^* and a stratification assignment σ_{P^*} for P^* in such a way that P results from P^* by substituting variables for the free variables (for example, P^* can be chosen as the term resulting from P when each free occurrence of a variable has been replaced by a fresh variable). Without loss of generality, we may suppose that the substitution that transforms P^* into P affects each free variable of P^*.

We define A^* and σ_{A^*}, for the weakly stratifiable formula A, in a similar way. We will write σ^* instead of σ_{P^*} or σ_{A^*}; the meaning will be clear from the context.

DEFINITION. Let v be a valuation into N, defined for P. $[\![P, v]\!]$ is $\pi^{-\sigma^*(P^*)}([\![P^*, \sigma^*, v^*]\!])$, where $v^*(x) = \pi^{\sigma^*(x)}(v(y))$, if y is substituted for x while converting P^* to P, and $v^*(x)$ is undefined if x is not free in P^* (σ^* is σ_{P^*}). □

Since $[\![P^*, \sigma^*, v^*]\!]$ belongs to $\mathrm{Adm}(\sigma^*(P^*))$ and π is an automorphism of $\mathfrak{M}$, $[\![P, v]\!]$ belongs to N.[12]

Now we let $\mathcal{N}$ be the set N with the function sort, the relations $\in_\Sigma$ and the operator $[\![\,,]\!]$ as defined above. Clearly $\mathcal{N}$ is a normalization structure. It remains to be shown that $\mathcal{N}$ is comprehensive, stable and sound.

Let us begin with comprehensiveness. Intuitively, comprehension should mean that $[\![\{x \mid A\}, v]\!]$ is the set of the ordered pairs (Σ, α) such that $\Sigma\downarrow, v' \Vdash A$ (for $v \prec_x v'$, $v'(x) = \alpha$ and α in N). On the other hand, $[\![\{x \mid A\}, v]\!]$ is an element of N. From the

[11] More exactly, N is a set canonically associated with Adm(0) and sort is the function of the real world corresponding to the object sort of $\mathfrak{M}$, which in $\mathfrak{M}$ is a "function" defined on Adm(0).

[12] The next lemma entails that the definition of $[\![P, v]\!]$ is independent of the particular choice of P^* and σ^*.

usual type-theoretic point of view, such a situation is impossible. So, $[\![\{x \mid A\}, v]\!]$ will not be this very set, but rather its code in N, as it is expressed in part 2 of the following lemma.

LEMMA. 1. $\Sigma, v \Vdash C$ *iff* $\Sigma, \sigma^*, v^* \Vdash C^*$;

2. $\pi([\![\{x \mid A\}, v]\!])$ *is the set of the ordered pairs* (Σ, α) *such that* $\Sigma\downarrow, v' \Vdash A$, *for* $v \prec_x v'$, $v'(x) = \alpha$ *and* α *in* N.

PROOF. If C is the formula $P \in Q$, then, using the fact that $\sigma^*(Q^*) = \sigma^*(P^*) + 1$, we have $\Sigma, \sigma^*, v^* \Vdash P^* \in Q^*$ iff

$$[\![P^*, \sigma^*, v^*]\!] \in_{\sigma^*(P^*), \Sigma} [\![Q^*, \sigma^*, v^*]\!]$$

iff

$$\pi^{-\sigma^*(P^*)}([\![P^*, \sigma^*, v^*]\!]) \in_{0, \Sigma} \pi^{-\sigma^*(P^*)}([\![Q^*, \sigma^*, v^*]\!])$$

iff $[\![P, v]\!] \in_{0,\Sigma} \pi([\![Q, v]\!])$ iff $\Sigma, v \Vdash P \in Q$.[13] One finishes by induction.

PROPOSITION 1. $\mathscr{N}$ *is comprehensive.*

PROOF. Let $v'(x) = \alpha$ and $v \prec_x v'$. By definition, $\alpha \in_\Sigma [\![\{x \mid A\}, v]\!]$ iff $\alpha \in_{0,\Sigma} \pi([\![\{x \mid A\}, v]\!])$, and, by part 2 of the lemma, $\alpha \in_{0,\Sigma} \pi([\![\{x \mid A\}, v]\!])$ iff $\Sigma\downarrow, v' \Vdash A$.

The substitution property is easily proved by induction (simultaneously with the first part of the substitution lemma) using the lemma and the fact that $[\![x, v]\!] = v(x)$.

PROPOSITION 2. $\mathscr{N}$ *is stable.*

This follows at once from the fact that the admissible sets are stable. □

PROPOSITION 3. $\mathscr{N}$ *is sound.*

PROOF. The cuts in a derivation Σ are naturally ordered, e.g. by considering the left to right order of their first symbol. This order induces a (finite) order on the derivations to which Σ reduces in one step. Therefore, if there are reduction sequences ofarbitrary finite length starting with Σ, we can safely define red(Σ) as the *first* derivation to which Σ reduces in one step and having reduction sequences of arbitrary finite length starting with it. Hence, if there are reduction sequences of arbitrary finite length starting with a derivation Σ, then the sequence Σ, red(Σ), red(red(Σ)), ... is an infinite reduction sequence. This shows (without invoking König's lemma) that a derivation is strongly normalizable iff there is a finite bound to the length of the reduction sequences starting from it.

The ω-standardness of $\mathfrak{M}$ then implies that a derivation is strongly normalizable in $\mathfrak{M}$ iff it is strongly normalizable (in the real world). The conclusion is now obvious. □

We have thus proved that $\mathscr{N}$ is a normalization model. Therefore the corollary to the main theorem implies the

NORMALIZATION THEOREM. *Every derivation of SF is strongly normalizable.*

Appendix. We will show that there exists a model $\mathfrak{M}$ of the theory $\mathfrak{T}$ having the properties assumed in §§4 and 5, namely ω-standardness, existence of elements u_i such that $\mathscr{P}\mathscr{P}\mathscr{P}(u_i) \subseteq u_{1+1}$ and existence of an automorphism π such that $\pi(u_i) = u_{i+1}$. The proof we give is a simplification of a similar result contained in [11].

Throughout this Appendix, we assume the axiom of choice or, alternatively,

[13] Here σ^* is σ_{C^*}.

we move into the constructible universe. We recall the definition of the beth operation: $\beth_0(\mu) = \mu$, $\beth_{\alpha+1}(\mu) = 2^{\beth_\alpha(\mu)}$ and $\beth_\lambda(\mu) = \bigcup_{\alpha<\lambda} \beth_\alpha(\mu)$ for λ limit. If X is a set and n a natural number, $[X]^n$ is the set of subsets of X having exactly n elements.

Henceforth, κ will denote $\beth_{\omega_1}$. Suppose that there is given a sequence of functions $f_1, f_2, f_3, \ldots$, each f_i in the sequence being a function from $[\kappa]^{\mathrm{ar}(f_i)}$ into ω; $\mathrm{ar}(f_i)$ is a natural number.

DEFINITIONS. Let H and G be subsets of K. H is said to be *n-indiscernible* from G (in notation: $H \equiv_n G$) iff every f_i is constant on $[H]^{\mathrm{ar}(f_i)} \cup [G]^{\mathrm{ar}(f_i)}$ $(1 \le i \le n)$.

Let $\mathscr{H}$ be a family of subsets of κ, none of them of size κ. $\mathscr{H}$ is *unbounded* iff

$$\forall \mu < \kappa \exists H \in \mathscr{H} \, |H| > \mu,$$

and $\mathscr{H}$ is *n-ambiguous* iff

$$\forall H, H' \in \mathscr{H} \; H \equiv_n H'.$$

$\mathscr{G}$ *extends* $\mathscr{H}$ iff $(\forall G \in \mathscr{G})(\exists H \in \mathscr{H})\; G \subseteq H$.

PROPOSITION. *If $\mathscr{H}$ is unbounded and n-ambiguous, there exists an unbounded $(n+1)$-ambiguous $\mathscr{G}$ extending $\mathscr{H}$.*

PROOF. 1. With the help of the Ramsey-Erdös-Rado theorem, one shows first that there is an unbounded $\mathscr{H}'$ extending $\mathscr{H}$ such that $H' \equiv_{n+1} H'$, if H' belongs to $\mathscr{H}'$. Let $\mu < \kappa$, μ infinite, and E a subset of cardinality $\beth_{\mathrm{ar}(f_{n+1})}(\mu)$ of some element of $\mathscr{H}$. Such an element exists because $\beth_{\mathrm{ar}(f_{n+1})}(\mu) < \kappa$ and $\mathscr{H}$ is unbounded. By the hypothesis and the Erdös-Rado theorem, there is an $H' \subseteq E$ such that $|H'| > \mu$, and $H' \equiv_{n+1} H'$. [14]

2. The cofinality of κ is greater than ω, and each f_i can take at most ω values. Therefore, there is an unbounded $(n+1)$-ambiguous subfamily of $\mathscr{H}'$. $\square$

Now we apply this proposition to the construction *à la* Ehrenfeucht-Mostowski of an ω-model of $\mathfrak{T}$ with indiscernibles. We add Skolem function symbols to the language of the predicate calculus with $\in$ and $=$ as nonlogical symbols and fix an interpretation of the Skolem symbols in V_κ (V_κ is the ZF-universe of sets up to κ) which makes the defining formulas for the Skolem functions true. We write $V_\kappa \models \phi$ iff V_κ satisfies ϕ for this interpretation.

Let $s_1, s_2, s_3, \ldots$ be an enumeration of the Skolem functions. To each s_i corresponds the function f_i from $[\kappa]^{\mathrm{ar}(s_i)}$ into ω defined as follows:

$$f_i(\{\alpha_1, \ldots, \alpha_{\mathrm{ar}(s_i)}\}) = s_i(V_{\alpha_1}, \ldots, V_{\alpha_{\mathrm{ar}(s_i)}})$$

if $s_i(V_{\alpha_1}, \ldots, V_{\alpha_{\mathrm{ar}(s_i)}})$ belongs to ω, and $f_i(\{\alpha_1, \ldots, \alpha_n\}) = 0$ else, for $\alpha_1 < \cdots < \alpha_{\mathrm{ar}(s_i)}$, $\mathrm{ar}(s_i)$ being the arity of s_i. Using the proposition, one constructs a sequence $\mathscr{H}_1, \mathscr{H}_2, \ldots$ of families of sets such that, for every n, $\mathscr{H}_n$ is unbounded and n-ambiguous, each of them, save $\mathscr{H}_1$, extending the previous one.

One then adds to the expanded language of $\mathfrak{T}$ a new set of individual constants c_i indexed by the integers. The set of sentences of the form $\phi(c_{i_1} \cdots c_{i_n})$, with

[14]The Erdös-Rado theorem says that if E is a set of a cardinality greater than $\beth_k(\mu)$, then for every function of $[E]^{k+1}$ into μ there is a subset F of E of cardinality greater than μ such that the function is constant on $[F]^{k+1}$; in standard notation: $\beth_k(\mu)^+ \to (\mu^+)^{k+1}_\mu$.

$i_1 < \cdots < i_n$, such that there is an increasing sequence of ordinals $\alpha_1, \ldots, \alpha_n$ in an element of some $\mathscr{H}_m$ ($m > k$, s_k being the Skolem function corresponding to the characteristic function of ϕ) with the property that $V_\kappa \models \phi[V_{\alpha_1} \cdots V_{\alpha_n}]$ forms a complete Skolem theory containing the sentence $\mathscr{P}\mathscr{P}\mathscr{P}(c_i) \subseteq c_{i+1}$.

Fix a model $\mathfrak{M}$ of this theory in which every object is denoted by a closed Skolem term. One naturally defines an automorphism π of $\mathfrak{M}$ by the following condition: if x is denoted by $t(c_{i_1} \cdots c_{i_n})$ then $\pi(x)$ is the object denoted by $t(c_{i_2} \cdots c_{i_n+1})$. Now letting, for each integer i, u_i be the element denoted by c_i, we have $\mathscr{P}\mathscr{P}\mathscr{P}(u_i) \subseteq u_{i+1}$ and $\pi(u_i) = u_{i+1}$, as needed.

It remains to be proved that the ω of $\mathfrak{M}$ is standard. Every element of ω is described by a closed Skolem term of the language that denotes an element of the ω of $\mathfrak{M}$. Conversely, if x is an element of $\mathfrak{M}$ such that $\mathfrak{M} \models x \in \omega$ and if $t(c_{i_1} \cdots c_{i_n})$, with $i_1 < \cdots < i_n$, denotes x, then the sentence $t(c_{i_1} \cdots c_{i_n}) \in \omega$ belongs to the theory. Therefore, there is a k such that if $m > k$ and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are increasing sequences of ordinals of (some elements H and G, respectively, of) $\mathscr{H}_m$, then

$$V_\kappa \models t[V_{\alpha_1} \cdots V_{\alpha_n}] \in \omega \quad \text{and} \quad V_\kappa \models t[V_{\alpha_1} \cdots V_{\alpha_n}] = t[V_{\beta_1} \cdots V_{\beta_n}].$$

If we set $f(x)$ equal to $t[V_{\alpha_1} \cdots V_{\alpha_n}]$, for one of these sequences, then f defines an isomorphism between the ω and $\mathfrak{M}$ and the real one.

REFERENCES

[1] S. C. Bailin, *A normalization theorem for set theory*, this Journal, vol. 53 (1988), pp. 673–695.

[2] N. Bourbaki, ***Éléments de mathématique***. Livre I: ***Théorie des ensembles***, 3rd ed., Hermann, Paris, 1958.

[3] M. Crabbé, *Ambiguity and stratification*, ***Fundamenta Mathematicae***, vol. 101 (1978), pp. 11–17.

[4] ———, *Typical ambiguity and the axiom of choice*, this Journal, vol. 49 (1984), pp. 1074–1078.

[5] H. B. Curry and R. Feys, ***Combinatory logic***, North-Holland, Amsterdam, 1968.

[6] N. G. de Bruijn, *Lambda calculus notation with nameless dummies, a tool for automatic manipulation, with application to the Church-Rosser theorem*, ***Indagationes Mathematicae***, vol. 34 (1972), pp. 391–392.

[7] J. Y. Girard, *Une extension de l'interprétation de Gödel à l'analyse, et son application á l'élimination des coupures dans l'analyse et la théorie des types*, ***Proceedings of the second Scandinavian logic symposium*** (J. E. Fenstad, editor), North-Holland, Amsterdam, 1971, pp. 63–92.

[8] ———, ***Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieur***, Thèse, Université Paris-VII, Paris, 1972.

[9] L. Hallnäs, ***On normalization of proofs in set theory***, Ph.D. thesis, University of Stockholm, Stockholm, 1983.

[10] W. Howard, *The formulae-as-types notion of construction*, ***To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*** (J. R. Seldin and J. R. Hindley, editors), Academic Press, New York, 1980, pp. 479–490.

[11] R. B. Jensen, *On the consistency of a slight (?) modification of NF*, ***Synthese***, vol. 19 (1968–69), pp. 250–263.

[12] P. Martin-Löf, *Hauptsatz for the theory of species*, ***Proceedings of the second Scandinavian logic symposium*** (J. E. Fenstad, editor), North-Holland, Amsterdam, 1971, pp. 217–233.

[13] D. Prawitz, ***Natural deduction***, Almqvist & Wiksell, Stockholm, 1965.

[14] ———, *Ideas and results in proof theory*, ***Proceedings of the second Scandinavian logic symposium*** (J. E. Fenstad, editor), North-Holland, Amsterdam, 1971, pp. 235–307.

[15] W. V. O. Quine, *New foundations for mathematical logic*, ***American Mathematical Monthly***, vol. 40 (1937), pp. 70–80; reprinted in ***From a logical point of view***, Harvard University Press, Cambridge, Massachusetts, 1953, pp. 80–101; rev. ed., Harper & Row, New York, 1961, pp. 80–101.

[16] E. Specker, *The axiom of choice in Quine's "New foundations for mathematical logic"*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 39 (1953), pp. 972–975.

[17] ———, *Typical ambiguity*, ***Logic, methodology and the philosophy of science*** (E. Nagel et al., editors), Stanford University Press, Stanford, California, 1962, pp. 117–124.

[18] W. W. Tait, *Intensional interpretation of functionals of finite type*, this Journal, vol. 32 (1967), pp. 198–212.

[19] ———, *A realizability interpretation of the theory of species*, ***Logic Colloquium (Boston, Massachusetts, 1972/73)***, Lecture Notes in Mathematics, vol. 453, Springer-Verlag, Berlin, 1975, pp. 240–251.

DÉPARTEMENT DE PHILOSOPHIE
UNIVERSITÉ DE LOUVAIN
B-1348 LOUVAIN-LA-NEUVE, BELGIUM