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TYPICAL AMBIGUITY AND THE AXIOM OF CHOICE

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E. Specker has proved that the axiom of choice (AC) is false in NF [6]. Since AC is stratified, one can, according to another famous result of Specker [7], prove directly $\neg$AC in type theory (TT) plus some finite set of ambiguity axioms, i.e. sentences of the form $\varphi \leftrightarrow \varphi^+$, where φ^+ results from φ by adding one to its type indices.

We shall in §2 of this paper give a disproof of AC directly in TT plus some axioms of ambiguity. The argument will be split into two parts. The first one (contained in Proposition 2) concerns cardinal arithmetic and has nothing to do with typical ambiguity. Though carried out in TT, it could have been done in other set theories such as Zermelo's Z or ZF. The second part is an application of this to the cardinals of the universes at different types. This is made possible through the introduction of an appropriate definition of 2^α in §1 enabling one to express shifting sentences as "typed properties" of the universe, in Boffa's sense. The disproof of AC is then completed in TT plus two extra ambiguity axioms. In §3, we show that this is in a sense the best possible result: that means that every single ambiguity axiom is *consistent* with TT plus AC, thus giving a positive solution to a conjecture of Specker [7, p. 119].

§1. The definition of 2^α. Until the end of §2 we work in TT. As usual, we omit the mention of type indices. V is the universe: $\{x \mid x = x\}$. Λ is the empty set: $\{x \mid x \neq x\}$. NC is the set of cardinal numbers. The letters $\alpha, \beta, \gamma, \delta$ will denote elements of NC. $|x|$ is the cardinal number of x; that is, the set of all sets equipollent to x. v is $|V|$. USC(x) is the set of all unit subsets of x: $\{\{y\} \mid y \in x\}$. SC(x) is the power set of x. $T|x|$ is $|\mathrm{USC}(x)|$. The inverse operation of T is defined by the clauses: $T^{-1}T\alpha = \alpha$ and $T^{-1}x = \Lambda$, if x is not a cardinal of the form $T\alpha$, i.e. $x \not\leq Tv$.

Now we want to define 2^α in such a way that it has the same type as α. This poses a problem, because $2^{|x|}$ is usually taken to be $|\mathrm{SC}(x)|$, which is located one type higher than $|x|$. One can adopt two different strategies to avoid this difficulty. The first one is to define $2^{|x|}$ just in case $|x| = |\mathrm{USC}(y)|$ for some y and then put $2^{|x|} = |\mathrm{SC}(y)|$. The second one consists in defining $2^{|x|}$ as $|y|$ when $|\mathrm{USC}(y)| = |\mathrm{SC}(x)|$, and leaving $2^{|x|} = \Lambda$ when there is no such y. The first definition was introduced by Specker and is usual in the TT-NF literature. The second one, which will be adopted here, has been introduced in [2], where it is shown to be more general, in TT and in NF, than

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the former. Formally one thus defines 2^α as follows:

$$2^{|x|} = T^{-1}|\mathrm{SC}(x)| \quad \text{and} \quad 2^x = \Lambda \text{ if } x \notin \mathrm{NC}.$$

Beth numbers may now be introduced as usual: $\beth_0(x) = x$, $\beth_{i+1}(x) = 2^{\beth_i(x)}$.

PROPOSITION 1. 1. $2^{|\mathrm{USC}(x)|} = |\mathrm{SC}(x)|$.

2. $2^\alpha \neq \Lambda \to T2^\alpha = 2^{T\alpha}$.

3. $2^{T\alpha} = T\beta \to 2^\alpha = \beta$.

4. $2^\alpha \neq \Lambda \to \alpha < 2^\alpha$ (*Cantor's theorem*).

PROOF. 1. $2^{|\mathrm{USC}(x)|} = T^{-1}|\mathrm{SC}(\mathrm{USC}(x))| = T^{-1}|\mathrm{USC}(\mathrm{SC}(x))| = |\mathrm{SC}(x)|$.

We let x be an element of α, i.e. $|x| = \alpha$.

2. Using 1, one obtains: $T2^\alpha = TT^{-1}|\mathrm{SC}(x)| = 2^{T\alpha}$, if $2^\alpha \neq \Lambda$.

3. Let $y \in \beta$. If $2^{T\alpha} = T\beta$, then, by 1, $|\mathrm{SC}(x)| = |\mathrm{USC}(y)|$. Hence $2^\alpha = T^{-1}|\mathrm{SC}(x)| = |y| = \beta$.

4. It is known that $|\mathrm{USC}(x)| < |\mathrm{SC}(x)|$. So $2^\alpha \neq \Lambda$ implies that $T\alpha < 2^{T\alpha} = T2^\alpha$, by 1 and 2. Thus $\alpha < 2^\alpha$. ■

Note that the third part of this proposition is not true for Specker's definition of 2^α.

We define $\Phi(\gamma, \alpha)$ as

$$\{\delta \in \mathrm{NC} \mid \forall x(\alpha \in x \wedge (\forall \beta \in x)(2^\beta \leq \gamma \to 2^\beta \in x) \to \delta \in x)\}$$

if $\alpha \leq \gamma$, and $\Phi(y, x) = \Lambda$ if $\{x, y\} \not\subseteq \mathrm{NC}$ or $x \not\leq y$. $\Phi(\gamma, \alpha)$ is thus the set $\{\alpha, 2^\alpha, 2^{2^\alpha}, \ldots\} \cap \{\delta \mid \delta \leq \gamma\}$. If γ is a cardinal number, we let γ_0 denote the least cardinal number β such that $\Phi(\gamma, \beta)$ is finite. More precisely $\gamma_0 = \beta$ if $\beta \leq \gamma$, $\Phi(\gamma, \beta)$ is finite and $(\forall \delta \leq \gamma)(\Phi(\gamma, \delta)$ is finite $\to \beta \leq \delta)$, and $\gamma_0 = \Lambda$ otherwise. We remark that $\Phi(\gamma, \gamma_0)$ is always finite and that $\Phi(\gamma, \gamma_0) \neq \Lambda$ iff $\gamma_0 \neq \Lambda$.

The formula $\gamma \in \Phi(\gamma, \gamma_0)$ means that γ is the last (the greatest) cardinal in $\Phi(\gamma, \gamma_0)$. We let Last(γ) denote this formula. We can also express that $\Phi(\gamma, \gamma_0)$ has an even number of elements by writing down the formula: "there is a partition of $\Phi(\gamma, \gamma_0)$ into two equipollent sets". Even(γ) will abreviate this formula.

Using Proposition 1, one proves readily that $\Phi(T\gamma, T\alpha) = \{T\delta \mid \delta \in \Phi(\gamma, \alpha)\}$ and that $(T\gamma)_0 = T\gamma_0$. So one obtains

LEMMA 0.1. Last(γ) $\leftrightarrow$ Last$^+(T\gamma)$.

2. Even(γ) $\leftrightarrow$ Even$^+(T\gamma)$.

REMARK. The meaning of this lemma becomes clear if one notices that Last($|v|$) and Even($|v|$) are typed properties in the sense of Boffa [1]. Indeed, if $v \neq \Lambda$ then Last($|v|$) says that the sentence Last(v) is true in the structure $\langle v, \mathrm{SC}(v), \mathrm{SC}(\mathrm{SC}(v)), \ldots \rangle$ which is "isomorphic" to $\langle \mathrm{USC}(v), \mathrm{SC}(\mathrm{USC}(v)), \mathrm{SC}(\mathrm{SC}(\mathrm{USC}(v))), \ldots \rangle$ (see [4]) and similarly for Even($|v|$).

§2. Let TT + AC be the theory of types with, as additional axioms, the sentences expressing the axiom of choice at each level. In the following AC will be used only through two of its consequences, namely $(\forall x \in \mathrm{NC})(x_0 \neq \Lambda)$ and $(\forall x \in \mathrm{NC})(\forall y \in \mathrm{NC})(x \leq y \wedge 2^x \neq \Lambda \to 2^x \leq y \vee y \leq 2^x)$.

LEMMA 1 (TT + AC). *If* $2^\gamma \neq \Lambda$, $\alpha \leq \gamma$ *and* $\Phi(\gamma, \alpha)$ *is finite, then*

$$\exists \beta(2^\beta \notin \Phi(\gamma, \alpha) \wedge (\Phi(2^\gamma, \alpha) = \Phi(\gamma, \alpha) \cup \{2^\beta\} \vee \Phi(2^\gamma, \alpha) = \Phi(\gamma, \alpha) \cup \{2^\beta, 2^\gamma\})).$$

PROOF. Assume the hypotheses. Since $\Phi(\gamma,\alpha)$ is finite and not empty, there is a greatest cardinal β in $\Phi(\gamma,\alpha)$. We have $2^\beta \not\leq \gamma$, and, with AC, this implies $\gamma < 2^\beta$. Thus, if $2^{2^\beta} \neq \Lambda$, $2^\beta \leq 2^\gamma \leq 2^{2^\beta}$. So $2^\beta \in \Phi(2^\gamma,\alpha)$. And $2^{2^\beta} \in \Phi(2^\gamma,\alpha)$ iff $2^{2^\beta} = 2^\gamma$.

LEMMA 2 (TT + AC). *If $2^\gamma \neq \Lambda$, then $\gamma_0 = (2^\gamma)_0$.*

PROOF. It follows from the hypothesis that $\Phi(\gamma,\gamma) = \{\gamma\}$ and $\Phi(2^\gamma,\gamma) = \{\gamma, 2^\gamma\}$. So, from AC, γ_0 and $(2^\gamma)_0$ are not empty and $(2^\gamma)_0 \leq \gamma$. Moreover, $\Phi(\gamma,(2^\gamma)_0) \subseteq \Phi(2^\gamma,(2^\gamma)_0)$. Thus, $\Phi(\gamma,(2^\gamma)_0)$ is finite and not empty, and $\gamma_0 \leq (2^\gamma)_0$. On the other hand, $\Phi(2^\gamma,\gamma_0)$ is also finite, by Lemma 1. Hence $(2^\gamma)_0 \leq \gamma_0$.

PROPOSITION 2 (TT + AC). 1. *If $2^\gamma \neq \Lambda$, then $\neg((\mathrm{Last}(\gamma) \leftrightarrow \mathrm{Last}(2^\gamma)) \wedge (\mathrm{Even}(\gamma) \leftrightarrow \mathrm{Even}(2^\gamma)))$.*

2. *If $2^{2^\gamma} \neq \Lambda$, then $\neg((\mathrm{Even}(\gamma) \leftrightarrow \mathrm{Even}(2^\gamma)) \wedge (\mathrm{Even}(2^\gamma) \leftrightarrow \mathrm{Even}(2^{2^\gamma})))$.*

PROOF. Let us suppose $2^\gamma \neq \Lambda$ and $\mathrm{Even}(\gamma) \leftrightarrow \mathrm{Even}(2^\gamma)$. By the two previous lemmas, we then have $\mathrm{Last}(2^\gamma)$. Hence, $2^{2^\gamma} \neq \Lambda$ entails $\Phi(2^{2^\gamma}, (2^\gamma)_0) = \Phi(2^\gamma,(2^\gamma)_0) \cup \{2^{2^\gamma}\}$ and so, by Lemma 2, $\mathrm{Even}(2^\gamma) \leftrightarrow \neg\, \mathrm{Even}(2^{2^\gamma})$. This proves 2.

Similarly, $\mathrm{Last}(\gamma)$ implies $\mathrm{Even}(\gamma) \leftrightarrow \neg\, \mathrm{Even}(2^\gamma)$, which proves 1. ■

If E is a formula or a term, E^+ results from E by raising the types by one. We shall use subsequently the well-known fact (see [7]) that $\mathrm{TT} \vdash \varphi$ entails $\mathrm{TT} \vdash \varphi^+$, for every formula φ.

THEOREM 1. *There are two sentences τ and σ such that the theories $TT + AC + (\tau \leftrightarrow \tau^+) + (\sigma \leftrightarrow \sigma^+)$ and $TT + AC + (\sigma \leftrightarrow \sigma^+) + (\sigma^+ \leftrightarrow \sigma^{++})$ are inconsistent.*

PROOF. The sentences τ and σ are $\mathrm{Last}(v)$ and $\mathrm{Even}(v)$, respectively. If we notice that $2^{Tv} = v^+$ and if we substitute Tv for γ in the first part of Proposition 2, we obtain

$$\neg((\mathrm{Last}^+(Tv) \leftrightarrow \tau^+) \wedge (\mathrm{Even}^+(Tv) \leftrightarrow \sigma^+))$$

as a theorem in TT + AC. We then use Lemma 0 to get the inconsistency of $\mathrm{TT} + \mathrm{AC} + (\tau \leftrightarrow \tau^+) + (\sigma \leftrightarrow \sigma^+)$.

Using the second part of Proposition 2 and the fact that $2^{TTv} = Tv^+$ and $2^{2^{TTv}} = v^{++}$, we obtain in a similar way the inconsistency of $\mathrm{TT} + \mathrm{AC} + (\sigma \leftrightarrow \sigma^+) + (\sigma^+ \leftrightarrow \sigma^{++})$. ■

Forster has proposed [3, pp. 59–61] to consider axioms of ambiguity of the form $\varphi \leftrightarrow \varphi^{+k}$, where φ^{+k} is $\varphi^{+\cdots+}$ (k times), $k > 0$. Translating Specker's original proof in TT, he notices that, given k, $3k$ axioms of this sort suffice to disprove AC. In fact two such axioms are enough. This will be shown now.

For each $k > 0$, one defines the sets $\varphi_k(\gamma,\alpha) = \{\alpha, \beth_k(\alpha), \beth_{2k}(\alpha), \ldots\} \cap \{\delta \mid \delta \leq \gamma\}$ by substituting $\beth_k(\beta)$ for 2^β in the definition of $\Phi(\gamma,\alpha)$ in §1. Remark that γ_0 as defined in §1 is also the least cardinal number α such that $\Phi_k(\gamma,\alpha)$ is finite and not empty. $\mathrm{Even}_k(\gamma)$ will be the formula expressing that $\Phi_k(\gamma,\gamma_0)$ has an even number of elements. Parallel to Lemmas 1 and 2 and Proposition 2, we have in TT + AC:

1. *$\beth_k(\gamma) \neq \Lambda$, $\alpha \leq \gamma$ and $\Phi_k(\gamma,\alpha)$ is finite imply $\Phi_k(\beth_k(\gamma),\alpha) = \Phi_k(\gamma,\alpha) \cup \{\beth_k(\beta)\}$ or $\Phi_k(\gamma,\alpha) \cup \{\beth_k(\beta), \beth_k(\gamma)\}$, for some $\beth_k(\beta)$ not in $\Phi_k(\gamma,\alpha)$.*

2. $(\beth_k(\gamma))_0 = \gamma_0$, *if $\beth_k(\gamma) \neq \Lambda$.*

3. *If $\beth_k(\gamma) \neq \Lambda$, then*

$$\neg((\mathrm{Last}(\gamma) \leftrightarrow \mathrm{Last}(\beth_k(\gamma))) \wedge (\mathrm{Even}_k(\gamma) \leftrightarrow \mathrm{Even}_k(\beth_k(\gamma))));$$

if $\beth_{2k}(\gamma) \neq \Lambda$, *then*

$$\neg((\mathrm{Even}_k(\gamma) \leftrightarrow \mathrm{Even}_k(\beth_k(\gamma))) \wedge (\mathrm{Even}_k(\beth_k(\gamma)) \leftrightarrow \mathrm{Even}_k(\beth_{2k}(\gamma)))).$$

PROOFS. The proof of 1 is an immediate generalization of the proof of Lemma 1, and 2 results by induction from Lemma 2. 3 is proved by generalizing the proof of Proposition 2; however, the last line becomes: Last(γ) implies $\beth_r(\beta) = \gamma$, for some $r < k$, and thus $\beth_{r+k}(\beta) = \beth_k(\gamma)$. The conclusion follows.

Let σ_k be the sentence $\mathrm{Even}_k(v)$. Theorem 1 generalizes to:

THEOREM 1 (BIS). *For each* $k > 0$, *the theories* $TT + AC + (\tau \leftrightarrow \tau^{+k}) + (\sigma_k \leftrightarrow \sigma_k^{+k})$ *and* $TT + AC + (\sigma_k \leftrightarrow \sigma_k^{+k}) + (\sigma_k^{+k} \leftrightarrow \sigma_k^{+2k})$ *are inconsistent.*

§3. The disproof of AC can be accomplished with two shifts: two sentences shifting once or one shifting twice. We show now that this is also necessary.

THEOREM 2. *For every sentence* φ *and natural number* $k > 0$, *the theory* $TT + AC + (\varphi \leftrightarrow \varphi^{+k})$ *is consistent.*

PROOF. Fix $k > 0$, φ and let $0, 1, \ldots, p-1$ contain the types occuring in φ. We may suppose that $p > 1$. By an abuse of notation, we identify φ with the stratifiable sentence of the language of ZF resulting from the omission of the type indices in φ and, if necessary, some changes of bound variables. We work in ZFC, where cardinals are as usual identified with initial ordinals. If α is a cardinal ($\alpha \neq 0$), let us write $\alpha \models \varphi$ for the formula $<\alpha, \mathscr{P}(\alpha), \ldots, \mathscr{P}^{p-1}(\alpha)> \models \varphi$ (here $\mathscr{P}$ is the power set operation of ZF, and the quantifiers of type i are restricted to $\mathscr{P}^i(\alpha)$ $(0 \leq i \leq p-1)$). If TT + AC + $(\varphi \leftrightarrow \varphi^{+k})$ were inconsistent, then ZFC would prove

$$\alpha \neq 0 \rightarrow (\alpha \models \varphi \leftrightarrow \beth_k(\alpha) \models \neg\varphi).$$

This is because $\langle \alpha, \mathscr{P}(\alpha), \ldots, \mathscr{P}^i(\alpha), \ldots \rangle$ is a model of TT + AC and φ, φ^{+k} are true within it just in case $\alpha \models \varphi$, $\beth_k(\alpha) \models \varphi$, respectively. Let us write $\alpha \sim \beta$ for "$\alpha \models \varphi \leftrightarrow \beta \models \varphi$". It will be sufficient to prove that

$$(*) \qquad \mathrm{ZFC} \vdash \alpha \neq 0 \rightarrow \neg\, \alpha \sim \beth_k(\alpha)$$

does not hold.

We suppose (*) true and derive a contradiction. n will denote henceforth the number $2(k+p) - 3$. Using forcing, we start with a countable transitive model $\mathbf{M}$ of ZFC having cardinals α_i $(0 \leq i \leq n)$ such that $\alpha_0 < \alpha_1 < \cdots < \alpha_n$, $2^{\alpha_i} = \alpha_{i+2}$, whenever $i < n-1$, $2^{\alpha_{n-1}} = \alpha_n$ and α_{n-1} is regular. The following diagram will illustrate this situation in case $k = 1$ and $p = 3$:

$$\begin{tikzcd}
 & \alpha_1 \arrow[r] & \alpha_3 \arrow[drr] & & \\
\alpha_0 \arrow[r] & \alpha_2 \arrow[rr] & & \alpha_4 \arrow[r] & \alpha_5
\end{tikzcd}$$

So, if i is odd and $i < n$, then there is a q such that $\beth_q(\alpha_i) = \beth_q(\alpha_{i+1})$, and it follows from (*) that $\alpha_i \sim \alpha_{i+1}$.

We use forcing again in order to collapse α_{n-1} and α_n without changing the situation below α_{n-1} (see [5]). This is done by extracting a generic set G from $\mathbb{C} = \{f \mid f$ is a function from a subset of α_{n-1} having power $< \alpha_{n-1}$ into $\alpha_n\}$. α_{n-1}

being regular, $\mathbb{C}$ is α_{n-1}-closed and, in $\mathbf{M}[G]$, α_i is a cardinal $(i \leq n-1)$, $2^{\alpha_i} = \alpha_{i+2}\,(i < n-2)$, and $2^{\alpha_{n-2}} = \alpha_{n-1}$. In case $k = 1$ and $p = 3$, the diagram above is thus changed into:

$$\begin{tikzcd}
 & \alpha_1 \arrow[rr] & & \alpha_3 \arrow[dr] & \\
\alpha_0 \arrow[rr] & & \alpha_2 \arrow[rr] & & \alpha_4
\end{tikzcd}$$

(*) implies that, in $\mathbf{M}[G]$, $\alpha_i \sim \alpha_{i+1}$ when i is even and $i < n-1$.

But, in $\mathbf{M}[G]$ there is no new subset of a set of $\mathbf{M}$ of power $< \alpha_{n-1}$. Thus, $\mathbf{M}[G] \models (\alpha_i \models \varphi)$ iff $\mathbf{M} \models (\alpha_i \models \varphi)$ $(i < 2k)$. So, going back to $\mathbf{M}$, we remark that $\alpha_0 \sim \alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_{2k-1} \sim \alpha_{2k}$. That is, $\alpha_0 \sim \beth_k(\alpha_0)$, contradicting (*). ∎

REMARKS. 1. The generalized continuum hypothesis can be disproved in TT plus one ambiguity axiom ($\sigma \leftrightarrow \sigma^+$ is such an axiom, though not the most natural one). For this reason the proof of Theorem 2 needs situations in which the GCH does not hold. Nevertheless it is compatible with the construction made in this proof that the GCH holds below α_0. In particular, every ambiguity axiom is consistent with TT + AC + CH.

2. One can be the forcing constructions within TT plus an axiom of infinity. Consequently, it is possible to weaken the assumption, made in the proof (for convenience), that ZF is consistent.

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