Cuts and gluts

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ABSTRACT. *We characterize the notion of validity relatively to models, for comprehension axioms, containing gluts.*

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1. Introduction

We will characterize the notion of validity with respect to models, for comprehension axioms, containing gluts. In order to make this issue clear, let us notice that the paper lies at the confluence of three rather independent topics that we now briefly mention.

1) Classical logic demands that a sentence is true iff it is not false, a requirement which encompasses both the excluded middle and the non-contradiction principle. Glut logic, on the contrary, only imposes that if a sentence is not true, then it is false, thus rejecting the non-contradiction principle, while keeping the excluded middle. So we can be confronted to situations where "not-true" may differ from "false", and "not-false" from "true". More on this topic can be found in [PRI 87].

2) Semantic proofs of cut-elimination for second order and higher order logics led people to a notion of model—more exactly of Schütte's semivaluation—which is reminiscent of the old three-valued logic (see [GIR 87, chapter 3]). Though these were presented in a context with gaps, one has the feeling that, on the semantic level, the removal of cuts becomes an introduction of gluts. It was also observed that a non-elementary cut-elimination proof is of no use in many interesting cases, because the cut-elimination can be proved in these cases almost trivially and in a very elementary way. This phenomenon is known as cut-absorbtion.

3) Following [GIL 74], different proposals were made for the solution of Russell's paradox, to the effect that one keeps the full naive comprehension principle in the

foundations of set theory, but changes the language by duplicating the $\in$-symbol. One will no longer have $r \in r$ iff $\neg r \in r$, but rather something like $r \in^+ r$ iff $r \in^- r$. To connect these two relations, one further imposes that there are no gaps $\neg(a \in^+ b \wedge a \in^- b)$ or no gluts $(a \in^+ b \vee a \in^- b)$, but not both (see [HIN 94], [LIB 04]).

We will be concerned here with cuts and gluts in systems for set theory, with abstraction terms. We will not use the two $\in$-symbols that we just mentioned, but we will rather bring this distinction on the semantic level. We will also not restrict ourselves to naive set theory, or some other particular system. We will instead define a semantics and a system of proofs which works for any language with abstracts. So our results will apply whether the system is consistent or not, or admits cut-elimination or not, etc.

A language will have exactly one binary relation symbol, denoted $\in$, and a collection of set abstracts $\{ x \mid \varphi \}$, closed under substitution. Thus the equality symbol is missing[1].

We will use individual constants instead of free variables. Formulas equivalent up to (bound) variables are identified. x, y, z will denote (bound) variables; a, b, c, d will denote individual constants; r, s, t, u will denote terms (constants or abstracts). We use the notation $\varphi(t)$ for highlighting some occurrences of t in φ. More precisely: the result of the substitution of t for c in φ is supposed to be denoted by $\varphi_c(t)$; since the mention of the subscript c will be useless—because clear from the context—in the situations we will be faced with, we use the notation $\varphi(t)$. Thus φ, which is $\varphi_c(c)$, is also denoted $\varphi(c)$.

2. Semantics

We introduce the notion of a model allowing gluts and define the truth and falsehood in such models. The main problem with set abstracts, in the general case, is that they block the inductive definitions. $t \in \{ x \mid \varphi(x) \}$ has not necessary greater complexity than $\varphi(t)$: consider, for example, $\{ x \mid x \in x \} \in \{ x \mid x \in x \wedge x \in x \}$. We will get round this difficulty by defining directly $t \in s$ from a predetermined interpretation of the terms, without worrying whether s is an abstract or not, and postulating independently afterwards a comprehension principle. The terms are interpreted as functions which assign values to valuations. We think this is better than interpreting abstracts as n-ary functions, what depends on arbitrary decisions, like an enumeration of the variables (more on this in [CRA 04b] and [CRA 04a]).

The definitions of this section are taken from [CRA 92]. Let $\mathcal{M} = \langle M, \in^+, \in^- \rangle$ be a structure such that M is not empty and $\in^+ \cup \in^- = M \times M$. A **valuation** is a

1. If the collection of abstracts is empty, we are left with the usual predicate calculus, without equality. The restriction on a single relation symbol is of course inessential—it simply makes things shorter to formulate and more readable.

function from the constants to M. $v[a \mapsto \alpha]$ is the valuation identical to v, except that the value of a is α, i.e., $v[a \mapsto \alpha](a) = \alpha$ and $v[a \mapsto \alpha](b) = v(b)$, for $b \neq a$. The **interpretation of a term** t is a function, denoted $\mathcal{M}(t)$, defined for the valuations, such that:

– $\mathcal{M}(t)(v)$ is in M;

– $\mathcal{M}(t)(v) = \mathcal{M}(t)(w)$ if $v(a) = w(a)$, for every constant a occuring in t;

– $\mathcal{M}(a)(v) = v(a)$, for every constant a;

– $\mathcal{M}(t(s))(v) = \mathcal{M}(t(a))(v[a \mapsto \mathcal{M}(s)(v)])$.

We define inductively $\mathcal{M}, v \models^+ \chi$ ("χ is true in $\mathcal{M}$ for v") and $\mathcal{M}, v \models^- \chi$ ("χ is false in $\mathcal{M}$ for v") as follows:

– $\mathcal{M}, v \models^+ t \in s$ iff $\mathcal{M}(t)(v) \in^+ \mathcal{M}(s)(v)$;

– $\mathcal{M}, v \models^- t \in s$ iff $\mathcal{M}(t)(v) \in^- \mathcal{M}(s)(v)$;

– $\mathcal{M}, v \models^+ (\varphi \to \psi)$ iff $\mathcal{M}, v \models^- \varphi$ or $\mathcal{M}, v \models^+ \psi$;

– $\mathcal{M}, v \models^- (\varphi \to \psi)$ iff $\mathcal{M}, v \models^+ \varphi$ and $\mathcal{M}, v \models^- \psi$;

– $\mathcal{M}, v \models^+ (\varphi \leftrightarrow \psi)$ iff $\mathcal{M}, v \models^+ \varphi$ and $\mathcal{M}, v \models^+ \psi$,
or $\mathcal{M}, v \models^- \varphi$ and $\mathcal{M}, v \models^- \psi$;

– $\mathcal{M}, v \models^- (\varphi \leftrightarrow \psi)$ iff $\mathcal{M}, v \models^+ \varphi$ and $\mathcal{M}, v \models^- \psi$,
or $\mathcal{M}, v \models^- \varphi$ and $\mathcal{M}, v \models^+ \psi$;

– $\mathcal{M}, v \models^+ \forall x\, \varphi(x)$ iff $\mathcal{M}, v[a \mapsto \alpha] \models^+ \varphi(a)$, for all α in M;

– $\mathcal{M}, v \models^- \forall x\, \varphi(x)$ iff $\mathcal{M}, v[a \mapsto \alpha] \models^- \varphi(a)$, for some α in M;

– $\mathcal{M}, v \models^+ \exists x\, \varphi(x)$ iff $\mathcal{M}, v[a \mapsto \alpha] \models^+ \varphi(a)$, for some α in M;

– $\mathcal{M}, v \models^- \exists x\, \varphi(x)$ iff $\mathcal{M}, v[a \mapsto \alpha] \models^- \varphi(a)$, for all α in M.

Such a structure $\mathcal{M}$, with an interpretation of the terms, is a **model** if it is **comprehensive**, i.e., if $\{\, x \mid \varphi \,\}$ is a term of the language, then for every α and v:

$\alpha \in^+ \mathcal{M}(\{\, x \mid \varphi \,\})(v)$ iff $\mathcal{M}, v[a \mapsto \alpha] \models^+ \varphi(a)$;
$\alpha \in^- \mathcal{M}(\{\, x \mid \varphi \,\})(v)$ iff $\mathcal{M}, v[a \mapsto \alpha] \models^- \varphi(a)$.

Given a structure, $\langle M, \in^+, \in^- \rangle$, an element of $\in^+ \cap \in^-$, if any, is called a **glut**.

A model is **classical** iff its underlying structure has no gluts.

The following standard properties hold also for this notion of model. The straightforward inductive proofs do not require comprehensiveness.

PROPOSITION 1. — *If $v(a) = w(a)$, for every constant in φ, then*
$\mathcal{M}, v \models^+ \varphi$ iff $\mathcal{M}, w \models^+ \varphi$, and $\mathcal{M}, v \models^- \varphi$ iff $\mathcal{M}, w \models^- \varphi$;
$\mathcal{M}, v[\vec{a} \mapsto \mathcal{M}(\vec{t})(v)] \models^+ \varphi(\vec{a})$ iff $\mathcal{M}, v \models^+ \varphi(\vec{t})$;
$\mathcal{M}, v[\vec{a} \mapsto \mathcal{M}(\vec{t})(v)] \models^- \varphi$ iff $\mathcal{M}, v \models^- \varphi(\vec{t})$;
if $\mathcal{M}, v \models^+ \forall x\, \varphi$, then $\mathcal{M}, v \models^+ \varphi(t)$;

if $\mathcal{M}, v \models^- \varphi(t)$*, then* $\mathcal{M}, v \models^- \forall x\, \varphi(x)$*;*
if $\mathcal{M}, v \models^+ \varphi(t)$*, then* $\mathcal{M}, v \models^+ \exists x\, \varphi(x)$*;*
if $\mathcal{M}, v \models^- \exists x\, \varphi(x)$*, then* $\mathcal{M}, v \models^- \varphi(t)$*.*

From comprehensiveness and proposition 1, we derive:

PROPOSITION 2. — $\mathcal{M}, v \models^+ t \in \{\, x \mid \varphi(x) \,\}$ *iff* $\mathcal{M}, v \models^+ \varphi(t)$*;*
$\mathcal{M}, v \models^- t \in \{\, x \mid \varphi(x) \,\}$ *iff* $\mathcal{M}, v \models^- \varphi(t)$*.*

3. The sequent calculus

A sequent is an ordered pair $\langle \Gamma, \Delta \rangle$ of finite sets of formulas, denoted $\Gamma \Vdash \Delta$. Γ, Γ' denotes $\Gamma \cup \Gamma'$, and $\{\varphi\}$ is denoted by φ.

CUT RULE. —

$$\frac{\Gamma \Vdash \varphi, \Delta \qquad \Gamma, \varphi \Vdash \Delta}{\Gamma \Vdash \Delta}$$

LOGICAL RULES. —

$$\frac{\Gamma \Vdash \varphi, \Delta}{\Gamma, \neg\varphi \Vdash \Delta}\,\neg_L \qquad\qquad \frac{\Gamma, \varphi \Vdash \Delta}{\Gamma \Vdash \neg\varphi, \Delta}\,\neg_R$$

$$\frac{\Gamma, \varphi, \psi \Vdash \Delta}{\Gamma, (\varphi \wedge \psi) \Vdash \Delta}\,\wedge_L \qquad\qquad \frac{\Gamma \Vdash \varphi, \Delta \qquad \Gamma \Vdash \psi, \Delta}{\Gamma \Vdash (\varphi \wedge \psi), \Delta}\,\wedge_R$$

$$\frac{\Gamma, \varphi \Vdash \Delta \qquad \Gamma, \psi \Vdash \Delta}{\Gamma, (\varphi \vee \psi) \Vdash \Delta}\,\vee_L \qquad\qquad \frac{\Gamma \Vdash \varphi, \psi, \Delta}{\Gamma \Vdash (\varphi \vee \psi), \Delta}\,\vee_R$$

$$\frac{\Gamma \Vdash \varphi, \Delta \qquad \Gamma, \psi \Vdash \Delta}{\Gamma, (\varphi \rightarrow \psi) \Vdash \Delta}\,\rightarrow_L \qquad\qquad \frac{\Gamma, \varphi \Vdash \psi, \Delta}{\Gamma \Vdash (\varphi \rightarrow \psi), \Delta}\,\rightarrow_R$$

$$\frac{\Gamma \Vdash \varphi, \psi, \Delta \quad \Gamma, \varphi, \psi \Vdash \Delta}{\Gamma, (\varphi \leftrightarrow \psi) \Vdash \Delta}\,\leftrightarrow_L \qquad\qquad \frac{\Gamma, \varphi \Vdash \psi, \Delta \qquad \Gamma, \psi \Vdash \varphi, \Delta}{\Gamma \Vdash (\varphi \leftrightarrow \psi), \Delta}\,\leftrightarrow_R$$

$$\frac{\Gamma, \varphi(t) \Vdash \Delta}{\Gamma, \forall x\, \varphi(x) \Vdash \Delta}\,\forall_L \qquad\qquad \frac{\Gamma \Vdash \varphi(a), \Delta}{\Gamma \Vdash \forall x\, \varphi(x), \Delta}\,\forall_R$$

$$\frac{\Gamma, \varphi(a) \Vdash \Delta}{\Gamma, \exists x\, \varphi(x) \Vdash \Delta}\,\exists_L \qquad\qquad \frac{\Gamma \Vdash \varphi(t), \Delta}{\Gamma \Vdash \exists x\, \varphi(x), \Delta}\,\exists_R$$

Usual restrictions for $\exists_L$, $\forall_R$ rules: the "proper" constant a doesn't appear in $\Gamma, \Delta, \varphi(x)$.

COMPREHENSION RULES. —

$$\frac{\Gamma, \varphi(t) \Vdash \Delta}{\Gamma, t \in \{\, x \mid \varphi(x) \,\} \Vdash \Delta}\{\,|\,\}_L \qquad \frac{\Gamma \Vdash \varphi(t), \Delta}{\Gamma \Vdash t \in \{\, x \mid \varphi(x) \,\}, \Delta}\{\,|\,\}_R$$

DEFINITIONS. — *A* ***prederivation*** *is a finite tree of sequents generated by the rules from* ***initial sequents****. More precisely: a sequent is a prederivation and its own initial sequent; if* $\overset{\vdots}{\Gamma'\Vdash\Delta'}$ *and* $\overset{\vdots}{\Gamma''\Vdash\Delta''}$ *are prederivations and if* $\dfrac{\Gamma'\Vdash\Delta' \quad \Gamma''\Vdash\Delta''}{\Gamma'\Vdash\Delta'}R$ *is an instance of the two-premiss rule* R*, then* $\dfrac{\overset{\vdots}{\Gamma'\Vdash\Delta'} \quad \overset{\vdots}{\Gamma''\Vdash\Delta''}}{\Gamma'\Vdash\Delta'}R$ *is a prederivation whose initial sequents are those of* $\overset{\vdots}{\Gamma'\Vdash\Delta'}$ *and* $\overset{\vdots}{\Gamma''\Vdash\Delta''}$*; and similarly for the one-premiss rules.*

$\Gamma \Vdash \Delta$ *is an* ***identity sequent*** *iff* $\Gamma \cap \Delta$ *is not empty.*

Let c*,* d *be distinct constants.* $\Gamma \Vdash \Delta$ *is a* cd***-sequent*** *iff* $c \in d$ *is in* Γ, Δ*.*

A ***derivation*** *is a prederivation whose initial sequents are identity sequents.*

If c*,* d *are distinct constants, a* cd***-derivation*** *is a cut-free prederivation, in which* c*,* d *are not proper constants and the initial sequents are identity or* cd*-sequents.*

A ***glut-derivation*** *is a* cd*-derivation—for some* c*,* d*—such that the constants* c*,* d *don't occur in the conclusion.*

WEAKENING. — If one adds Γ' to the left and Δ' to the right of each sequent in a prederivation of $\Gamma \Vdash \Delta$ while changing the proper constants to avoid conflicts, if necessary, then one gets a prederivation of $\Gamma', \Gamma \Vdash \Delta, \Delta'$. Clearly this weakening of a prederivation preserves its length and its kind (derivation, cut-free derivation, cd-derivation).

PROPOSITION 3 (INVERSION). — *The rules* $\{\,|\,\}_L$ *and* $\{\,|\,\}_R$ *are invertible in the following sense: if* $\Gamma, t \in \{\, x \mid \varphi \,\} \Vdash \Delta$ *is cut-free derivable, then* $\Gamma, \varphi(t) \Vdash \Delta$ *is cut-free derivable; if* $\Gamma \Vdash t \in \{\, x \mid \varphi \,\}, \Delta$ *is cut-free derivable, then* $\Gamma \Vdash \varphi(t), \Delta$ *is cut-free derivable. This holds also for* cd*-derivability, glut derivability and derivability.*

All the rules, except $\forall_L$ *and* $\exists_R$*, are invertible in an analogous sense.*

PROOF. — By induction on the length of a [cd-]derivation of $\Gamma, t \in \{\, x \mid \varphi \,\} \Vdash \Delta$.

If $\Gamma, t \in \{\, x \mid \varphi(x) \,\} \Vdash \Delta$ is an identity sequent, but $\Gamma, \varphi(t) \Vdash \Delta$ is not, then $t \in \{\, x \mid \varphi(x) \,\}$ belongs to Δ, and $\Gamma, \varphi(t) \Vdash \Delta$ is cd-derived as follows:

$$\frac{\Gamma, \varphi(t) \Vdash \varphi(t), \Delta}{\Gamma, \varphi(t) \Vdash \Delta}\{\,|\,\}_R$$

If $\Gamma, t \in \{\, x \mid \varphi(x) \,\} \Vdash \Delta$ is a cd-sequent, so is $\Gamma, \varphi(t) \Vdash \Delta$.

Suppose that the [cd-]derivation of $\Gamma, t \in \{\, x \mid \varphi(x)\,\} \Vdash \Delta$ ends in an inference introducing $t \in \{\, x \mid \varphi(x)\,\}$:

$$\frac{\Gamma^*, \varphi(t) \Vdash \Delta}{\Gamma^*, t \in \{\, x \mid \varphi(x)\,\} \Vdash \Delta}\ \{\,|\,\}_L,$$

where $\Gamma^*, t \in \{\, x \mid \varphi(x)\,\}$ is $\Gamma, t \in \{\, x \mid \varphi(x)\,\}$. Since we can suppose $t \in \{\, x \mid \varphi(x)\,\}$ not in Γ, we have $\Gamma^* = \Gamma$ or $\Gamma^* = \Gamma, t \in \{\, x \mid \varphi(x)\,\}$.

If $\Gamma = \Gamma^*$, the result is immediate from the premiss. If $\Gamma, t \in \{\, x \mid \varphi(x)\,\} = \Gamma^*$, the premiss is $\Gamma, t \in \{\, x \mid \varphi(x)\,\}, \varphi(t) \Vdash \Delta$, and the conclusion follows by inductive hypothesis.

If $\Gamma, t \in \{\, x \mid \varphi(x)\,\} \Vdash \Delta$ is the conclusion of an inference not introducing $t \in \{\, x \mid \varphi(x)\,\}$, then we use the same rule with premisses provided by the inductive hypothesis.

The invertibility of the $\{\,|\,\}_R$-rule is proved in a dual way.

Similar proofs work for the other rules; we can also deduce it from the completeness theorem (theorem 4), whose proof uses the invertibility of the $\{\,|\,\}$-rules only. ■

DEFINITIONS. — *A sequent $\Gamma \Vdash \Delta$ is **valid in** $\mathcal{M}$ **with respect to** v iff some formula in Γ is false in $\mathcal{M}$ with respect to v, or some formula in Δ is true in $\mathcal{M}$ with respect to v.*

*A sequent is **valid in** $\mathcal{M}$ iff it is valid in $\mathcal{M}$ with respect to every valuation.*

*A sequent is **valid**, **classically valid**, or **glut valid** iff it is valid in all models, all classical models, or all models with gluts, respectively.*

THEOREM 4 (COMPLETENESS). — *A sequent is valid, classically valid, or glut valid iff it is cut-free derivable, derivable, or glut derivable, respectively.*

PROOF. — We present the proof for the glut-case only. The two other cases can be easily obtained by an appropriate modification. In fact the classical case is the usual completeness theorem and the general case is practically in [CRA 92]. Besides, most of this proof is an adaptation of similar arguments that can be found in [CRA 92] and [CRA 94].

1. For the "if" part we will actually establish the following stronger result, as we will need it in the "only if" part: *a cd-derivable sequent is valid relatively to abstract models with gluts*—a model being called "abstract" when comprehension is relaxed to:

$$\text{if } \mathcal{M}, v[a \mapsto \alpha] \models^{\pm} \varphi(a), \text{ then } \alpha \in^{\pm} \mathcal{M}(\{\, x \mid \varphi\,\})(v).$$

Let indeed $\mathcal{M}$ be such a model, and α, β elements of M such that $\alpha \in^+ \beta$ and $\alpha \in^- \beta$. By induction on the length of a cd-derivation—using propositions 1, 2—every sequent is valid in $\mathcal{M}$ with respect to any valuation v such that $v(c) = \alpha$ and

$v(d) = \beta$. Therefore, as c, d don't occur in the conclusion, this conclusion is valid in $\mathcal{M}$ for all valuations.[2]

2. For the "only if" part, we give a Henkin-style argument.

Suppose that $\Gamma \Vdash \Delta$ is not glut derivable. Let c, d be such that it is not cd-derivable. We will show that there is a model with gluts and a valuation that makes $\Gamma \Vdash \Delta$ not valid.

2.1. We first indicate how to construct a sequence $\mathcal{G}$ of sequents $\Gamma_0 \Vdash \Delta_0, \Gamma_1 \Vdash \Delta_1, \ldots$ such that:

– no sequent in the sequence is cd-derivable; $\Gamma_0 \Vdash \Delta_0$ is $\Gamma \Vdash \Delta$; $\Gamma_i \subseteq \Gamma_{i+1}$ and $\Delta_i \subseteq \Delta_{i+1}$;

– for each formula of the form $t \in s$, if no $\Gamma_i \Vdash t \in s, \Delta_i$ is cd-derivable, then $t \in s$ is in some Δ_j; for each formula of the form $\forall x\, \psi(x)$, if no $\Gamma_i \Vdash \forall x\, \psi(x), \Delta_i$ is cd-derivable, then there is a term t such that $\psi(t)$ is in some Δ_j; for each formula of the form $\exists x\, \psi(x)$, if no $\Gamma_i, \exists x\, \psi(x) \Vdash \Delta_i$ is cd-derivable, then there is a term t such that $\psi(t)$ is in some Γ_j.

Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of the formulas of the language. We show how to go from $\Gamma_i \Vdash \Delta_i$ to $\Gamma_{i+1} \Vdash \Delta_{i+1}$.

If φ_i is $t \in s$ and $\Gamma_i \Vdash t \in s, \Delta_i$ is not cd-derivable, we let $\Gamma_{i+1} \Vdash \Delta_{i+1}$ be $\Gamma_i \Vdash t \in s, \Delta_i$. If φ_i is $\exists x\, \psi(x)$ and $\Gamma_i, \exists x\, \psi(x) \Vdash \Delta_i$ is not cd-derivable, let a be a constant not occuring in one of the formulas of this sequent and *distinct from c, d*. We let $\Gamma_{i+1} \Vdash \Delta_{i+1}$ be $\Gamma_i, \psi(a) \Vdash \Delta_i$. The analogous $\forall x\, \psi$ case is symmetric. In all the other cases we let $\Gamma_{i+1} \Vdash \Delta_{i+1}$ be $\Gamma_i \Vdash \Delta_i$.

2.2. We derive from this sequence the structure $\mathcal{M}$ as follows:

– M is the set of all terms—including the "open[3]" ones—of the language;

– $\mathcal{M}(\in)^+$ is the set of ordered pairs of terms $\langle t, s\rangle$ such that, for some i, $\Gamma_i \Vdash t \in s, \Delta_i$ is cd-derivable; and $\mathcal{M}(\in)^-$ is the set of ordered pairs of terms $\langle t, s\rangle$ such that, for some $i, \Gamma_i, t \in s \Vdash \Delta_i$ is cd-derivable;

– $\mathcal{M}(t(\vec{a}))(v)$ is $t(v(\vec{a}))$, for all $t(\vec{a})$; all the constants occuring in t being mentioned in $\vec{a}$.

2.3. $\in^+ \cup \in^- = M^2$, because if $t \not\in^+ s$, then no $\Gamma_i \Vdash t \in s, \Delta_i$ is cd-derivable. Hence $t \in s$ belongs to a Δ_j. Therefore $\Gamma_j, t \in s \Vdash \Delta_j$ is an identity sequent and $t \in^- s$.

2. We notice that the cut rule retains classical validity, but not validity nor glut validity.
3. So this is not a term model in the sense of [FOR 95].

– $\mathcal{M}$ has at least one glut, as we trivially have $c \in^+ d$ and $c \in^- d$.

2.4. We define $\mathcal{G} \models^+ \varphi$ as meaning that, for some i, $\Gamma_i \Vdash \varphi, \Delta_i$ is cd-derivable; and $\mathcal{G} \models^- \varphi$ as meaning that, for some i, $\Gamma_i, \varphi \Vdash \Delta_i$ is cd-derivable.

Let id be the identity valuation: $v(a) = a$, for every constant a. Clearly,

$$\mathcal{M}(t)(id) = t.$$

We now show that;

– if $\mathcal{M}, id \models^+ \varphi$, then $\mathcal{G} \models^+ \varphi$;

– if $\mathcal{M}, id \models^- \varphi$, then $\mathcal{G} \models^- \varphi$.

The proof is an induction on φ, starting with the immediate cases of the form $t \in s$; the "length" of $\psi(t)$ being less than the one of $\forall x\, \psi(x)$. We only give some cases.

– If $\mathcal{M}, id \models^+ (\psi \to \chi)$, then $\mathcal{M}, id \models^- \psi$ or $\mathcal{M}, id \models^+ \chi$. By inductive hypothesis, $\mathcal{G} \models^- \psi$ or $\mathcal{G} \models^+ \chi$. By weakening, $\Gamma_i, \psi \Vdash \chi, \Delta_i$ is cd-derivable, for some i. Therefore, applying the $\to_R$ rule, $\Gamma_i \Vdash (\psi \to \chi), \Delta_i$ is cd-derivable. Hence $\mathcal{G} \models^+ (\psi \to \chi)$.

– If $\mathcal{M}, id \models^- (\psi \to \chi)$, then $\mathcal{M}, id \models^+ \psi$ and $\mathcal{M}, id \models^- \chi$. By inductive hypothesis, $\mathcal{G} \models^+ \psi$ and $\mathcal{G} \models^- \chi$. Hence, by weakening, $\Gamma_i \Vdash \psi, \Delta_i$ and $\Gamma_i, \chi \Vdash \Delta_i$ are cd-derivable, for some i. Thus $\Gamma_i, (\psi \to \chi) \Vdash \Delta_i$ is cd-derivable, by the $\to_L$ rule. Whence $\mathcal{G} \models^- (\psi \to \chi)$.

– If $\mathcal{G} \not\models^+ \forall x\, \psi(x)$, then some $\psi(t)$ is in a Δ_i, which implies, by weakening, that $\mathcal{G} \not\models^+ \psi(t)$. By inductive hypothesis, $\mathcal{M}, id \not\models^+ \psi(t)$. Therefore $\mathcal{M}, id \not\models^+ \forall x\, \psi(x)$.

– If $\mathcal{M}, id \models^- \forall x\, \psi(x)$, then $\mathcal{M}, id[a \mapsto t] \models^- \psi(a)$, for some t; hence $\mathcal{M}, id \models^- \psi(t)$, as $\mathcal{M}(t)(id) = t$. By inductive hypothesis, $\Gamma_i, \psi(t) \Vdash \Delta_i$ is cd-derivable, for some i. And so is $\Gamma_i, \forall x\, \psi(x) \Vdash \Delta_i$, by the $\forall_L$ rule. Therefore $\mathcal{G} \models^- \forall x\, \psi(x)$.

From this we conclude that no sequent of $\mathcal{G}$ is valid in $\mathcal{M}$ with respect to id.

2.5. At last, to conclude that $\Gamma \Vdash \Delta$ is not glut valid, it remains to show that $\mathcal{M}$ is comprehensive.

We notice that, starting from the fact that $\mathcal{M}(t(\vec{b}))(v) = t(v(\vec{b})) = \mathcal{M}(t(v(\vec{b})))(id)$, one obtains, by induction, that $\mathcal{M}, v \models^\pm \psi(\vec{b})$ iff $\mathcal{M}, id \models^\pm \psi(v(\vec{b}))$, where all the constants occuring in ψ are indicated. So it will be sufficient to show that:

– if $\mathcal{M}, id \models^\pm \varphi(t)$, then $t \in^\pm \{\, x \mid \varphi(x) \,\}$, i.e., $\mathcal{M}$ is abstract,

– if $t \in^\pm \{\, x \mid \varphi(x) \,\}$, then $\mathcal{M}, id \models^\pm \varphi(t)$.

We limit ourselves to the $\models^+$ case, the other case being completely symmetric.

2.5.1. Suppose $\mathcal{M}, id \models^+ \varphi(t)$. Then, by 2.4., $\mathcal{G} \models^+ \varphi(t)$. Hence some $\Gamma_i \Vdash \varphi(t), \Delta_i$ is cd-derivable. So, by the $\{\,|\,\}_R$-rule, $\Gamma_i \Vdash t \in \{\, x \mid \varphi(x)\,\}, \Delta_i$ is also cd-derivable. Therefore $t \in^+ \{\, x \mid \varphi(x)\,\}$.

2.5.2. Suppose $t \in^+ \{\, x \mid \varphi(x)\,\}$. Then some $\Gamma_i \Vdash t \in \{\, x \mid \varphi(x)\,\}, \Delta_i$ is cd-derivable. By proposition 3, $\Gamma_i \Vdash \varphi(t), \Delta_i$ is also cd-derivable. Since $\mathcal{M}$ is abstract, by 2.5.1., it follows, from the proof of the "if" part, that $\Gamma_i \Vdash \varphi(t), \Delta_i$ is valid in $\mathcal{M}$ with respect to id. But, by 2.4., no formula in Γ_i is false and no formula in Δ_i is true in $\mathcal{M}$ with respect to id. Therefore $\mathcal{M}, id \models^+ \varphi(t)$.[4] ■

The following consequences of the completeness theorem are worth mentioning. If cut-elimination holds, valid means classically valid. If cut-elimination does not hold, then some classically valid sequent is not glut valid. Since $\Gamma \Vdash \Delta$ is valid iff it is glut valid AND classically valid, it follows that $\Gamma \Vdash \Delta$ is cut-free derivable iff it is glut derivable AND derivable. This observation will be proved directly and refined in the following theorem and its corollary. An analogous result for second order logic is already in [GIR 87].

THEOREM 5. — *If* $\Gamma \Vdash \Delta$ *is derivable and* $\Gamma^* \Vdash \Delta^*$ *is glut derivable, then* $\Gamma, \Gamma^* \Vdash \Delta^*, \Delta$ *is cut-free derivable. In fact, a cut-free derivation of* $\Gamma, \Gamma^* \Vdash \Delta^*, \Delta$ *can be produced in an elementary way from a given derivation of* $\Gamma \Vdash \Delta$ *and a given glut-derivation of* $\Gamma^* \Vdash \Delta^*$.

PROOF. — It might first be useful to motivate the proof by the following paradigmatic example. From a derivation of $\Gamma \Vdash \Delta$ with atomic cuts, we get a cut-free derivation of $\Gamma, \forall x \forall y\,(x \in y \to x \in y) \Vdash \Delta$: we simply add $\forall x \forall y\,(x \in y \to x \in y)$ to the left of each sequent and replace each cut

$$\frac{\Gamma', \Vdash t \in s, \Delta' \quad \Gamma', t \in s \Vdash \Delta'}{\Gamma' \Vdash \Delta'}$$

by

$$\dfrac{\dfrac{\Gamma' \Vdash t \in s, \Delta' \quad \Gamma', t \in s \Vdash \Delta'}{\Gamma', (t \in s \to t \in s) \Vdash \Delta'}\,\to_L}{\overline{\overline{\Gamma' \Vdash \Delta'}}}\,\forall_L, \forall_L$$

The proof below is the generalisation of the fact that we *absorbed* the cuts, in this example, by using the glut-derivation:

$$\dfrac{\dfrac{\Vdash c \in d \quad c \in d \Vdash}{(c \in d \to c \in d) \Vdash}\,\to_L}{\overline{\overline{\forall x \forall y\,(x \in y \to x \in y) \Vdash}}}\,\forall_L, \forall_L$$

4. Gilmore-Kripke's fixed-point construction could have been used in this paragraph instead of the inversion of the rules for $\{\,|\,\}$ (see [CRA 92]).

Choose a derivation of $\Gamma \Vdash \Delta$ and use a standard elementary cut-elimination procedure to reduce the cut-formulas (if any) to formulas of the form $t \in s$. Then add Γ^* to the left and Δ^* to the right of each sequent—and modify the proper constants if necessary.

We can now eliminate the cuts from the modified derivation, with the help of a glut-derivation of $\Gamma^* \Vdash \Delta^*$:

$$\begin{array}{c} \Gamma_i(c,d) \Vdash c \in d, \Delta_i(c,d) \;\cdots\; \Gamma_j(c,d), c \in d \Vdash \Delta_j(c,d) \\ \vdots \\ \Gamma^* \Vdash \Delta^* \end{array}$$

that we will properly attenuate. To this effect we select an uppermost cut

$$\begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \Sigma \Vdash t \in s, \Pi\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma, t \in s \Vdash \Pi\end{array}}{\Sigma \Vdash \Pi} \\ \vdots \\ \Gamma, \Gamma^* \Vdash \Delta^*, \Delta \end{array}$$

and replace it by

$$\begin{array}{c} \begin{array}{c}\vdots\\ \Sigma, \Gamma_i(t,s) \Vdash t \in s, \Delta_i(t,s), \Pi\end{array} \;\cdots\; \begin{array}{c}\vdots\\ \Sigma, \Gamma_j(t,s), t \in s \Vdash \Delta_j(t,s), \Pi\end{array} \\ \vdots \\ \Sigma \Vdash \Pi \\ \vdots \\ \Gamma, \Gamma^* \Vdash \Delta^*, \Delta \end{array}$$

thus obtaining a derivation with one less cut. ■

COROLLARY 6. — *From a derivation and a glut-derivation of a given sequent one obtains, in an elementary way, a cut-free derivation of the same sequent.*

REMARK. — If the empty sequent $\Vdash$ is derivable, then a sequent is cut-free derivable iff it is glut derivable. This happens, in particular, if the language contains the term $r : \{\, x \mid x \notin x \,\}$. Russell's paradox can indeed be presented as a derivation of $\Vdash$:

$$\dfrac{\dfrac{\dfrac{r \in r \Vdash r \in r}{\Vdash r \notin r, r \in r}\neg_R}{\Vdash r \in r}\{\,|\,\}_R \qquad \dfrac{\dfrac{r \in r \Vdash r \in r}{r \in r, r \notin r \Vdash}\neg_L}{r \in r \Vdash}\{\,|\,\}_L}{\Vdash}\mathit{cut}$$

□

3.1. *Loose derivations*

Let us now consider languages containing a set abstract $\boxtimes$ such that in a model with gluts every element bears the $\in^+$ and also the $\in^-$ relation to the interpretation of $\boxtimes$. An empty set abstract like $\{\, x \mid \exists x \exists y\, (x \in y \wedge x \notin y) \,\}$ will do[5].

DEFINITIONS. — $\Gamma \Vdash \Delta$ *is a d-**sequent** iff some formula of the form $t \in d$ belongs to* Γ, Δ.
*A **loose derivation** is a cut-free derivation in which the initial sequents are identity sequents or d-sequents, provided these d's are not proper constants and don't occur in the conclusion.*

PROPOSITION 7. — *If the language contains $\boxtimes$, then a sequent is glut derivable iff it is loosely derivable.*

PROOF. — Clearly every glut-derivation is loose.

Suppose there is a loose derivation of $\Gamma \Vdash \Delta$. We first replace the d's of the d-sequents by $\boxtimes$ and, in the resulting prederivation, the non-identity initial sequents of the form $\Gamma', t \in \boxtimes \Vdash \Delta'$ and $\Gamma' \Vdash t \in \boxtimes, \Delta'$ by the following glut-derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma', a \in b \Vdash a \in b, \Delta'}{\Gamma', a \in b, a \notin b \Vdash \Delta'}\,\neg_L}{\Gamma', (a \in b \wedge a \notin b) \Vdash \Delta'}\,\wedge_L}{\Gamma', \exists x \exists y\, (x \in y \wedge x \notin y) \Vdash \Delta'}\,\exists_L, \exists_L}{\Gamma', t \in \boxtimes \Vdash \Delta'}\,\{\,|\,\}_L
\qquad
\dfrac{\dfrac{\dfrac{\Gamma' \Vdash c \in d, \Delta' \quad \dfrac{\Gamma', c \in d \Vdash \Delta'}{\Gamma' \Vdash c \notin d, \Delta'}\,\neg_R}{\Gamma' \Vdash (c \in d \wedge c \notin d), \Delta'}\,\wedge_R}{\Gamma \Vdash \exists x \exists y\, (x \in y \wedge x \notin y), \Delta'}\,\exists_R, \exists_R}{\Gamma' \Vdash t \in \boxtimes, \Delta'}\,\{\,|\,\}_R$$

■

From proposition 7 and theorem 4, we obtain the

COROLLARY 8. — *If the language contains $\boxtimes$, a sequent is glut valid iff it is loosely derivable.*

3.2. *Cut absorption*

The two next notions come from [GIR 87], chapter 3, where they are introduced to minimize the importance of Takeuti's conjecture.

DEFINITIONS. — *A formula is **cut-absorbing** iff it is false in every model with gluts, with respect to every valuation.*

*A formula is **cut-proof** iff it is is true in every model with gluts, with respect to every valuation.*

5. Such an abstract was used in [HIN 87] in the gap-case. A universal set abstract like $\{\, x \mid \forall x \forall y\, (x \in y \to x \in y) \,\}$ could have been taken instead. Every empty set won't work though. $\{\, x \mid \forall x \forall y (x \in y \wedge x \notin y) \,\}$ is empty in every model but not universal in most of them.

Thus φ is cut-absorbing iff the sequent $\varphi \Vdash$ is glut valid, and φ is cut-proof iff the sequent $\Vdash \varphi$ is glut valid. Therefore, by theorem 4, φ is cut-absorbing iff the sequent $\varphi \Vdash$ is glut derivable, and φ is cut-proof iff the sequent $\Vdash \varphi$ is glut derivable.

Suppose that we can prove that φ is cut-absorbing and not cut-free derivable, then, by corollary 6, the (classical) consistency of φ follows *in an elementary way*. So if φ is cut-absorbing and if it is known that its consistency is not elementary provable, then it seems hopeless to rest on a cut-elimination theorem, like the one in [CRA 94], for showing in an elementary way that $\varphi \Vdash$ is not derivable.

A simple semantic argument, or proposition 3, shows that:

– φ is cut-proof iff $\neg\varphi$ is cut-absorbing; φ is cut-absorbing iff $\neg\varphi$ is cut-proof.

– $\varphi \wedge \psi$ is cut-proof iff φ and ψ are cut-proof.

– $\varphi \vee \psi$ is cut-absorbing iff φ and ψ are cut-absorbing.

– $\varphi \to \psi$ is cut-absorbing iff φ is cut-proof and ψ is cut-absorbing.

– $\forall x\, \varphi(x)$ is cut-proof iff $\varphi(a)$ is cut-proof; therefore φ is cut-proof iff its universal closures are cut-proof.

– $\exists x\, \varphi(x)$ is cut-absorbing iff $\varphi(a)$ is cut-absorbing; therefore φ is cut-absorbing iff its existential closures are cut-absorbing.

EXAMPLES. — $\exists x \exists y\, x \in y$, $\exists x \exists y\, x \notin y$, $\exists x \exists y \exists z\, (x \in y \wedge y \notin z)$ are cut-proof; and $\forall x \forall y\, x \in y$, $\forall x \forall y\, x \notin y$, $\forall x \forall y \forall z\, (x \in y \to y \in z)$ are cut-absorbing.

– $\forall x \forall y\, (x \in y \to x \in y)$ and $\exists x \exists y\, (x \in y \wedge x \notin y)$ are both cut-absorbing and cut-proof.

– $\exists x \forall y\, y \in x$, $\exists x \forall y\, (y \in x \wedge y \notin x)$ are cut-absorbing and cut-proof if the language contains $\otimes$, but not in general.

– "$\in$ is reflexive", viz $\forall x\, x \in x$ and "$\in$ is transitive", viz $\forall x \forall y \forall z\, (x \in y \wedge y \in z \to x \in z)$ are not generally cut-absorbing, but they are so if the language contains $\otimes$; "$\in$ is reflexive and transitive" is always cut-absorbing.

– The following "axiom of infinity" is cut-proof if the language contains $\otimes$:

$$\exists x\, (\emptyset \in x \wedge \forall z\, (z \in x \to \mathrm{Succ}(z) \in x) \wedge \exists y\, y \notin x),$$

where $\emptyset$ is any "emptyset" abstract and $\mathrm{Succ}(z)$ is any term—intuitively standing for the "successor" of z. To see it quickly, just take $\otimes$ as a witness for x.

$\square$

3.3. *Extensionality*

Let $a \mathrel{\dot\subset} b$ abbreviate $\forall x\, (a \in x \to b \in x)$ and $a \sim b$ be short for $\forall x\, (x \in a \leftrightarrow x \in b)$. *Ext* is the sentence $\forall x \forall y\, (x \sim y \to x \mathrel{\dot\subset} y)$.

As $\forall x \forall y\,(x \sim y \leftrightarrow y \sim x)$ is a logical truth, *Ext* is equivalent to $\forall x \forall y\,(x \sim y \to \forall z\,(x \in z \leftrightarrow y \in z))$. Thus *Ext* stands generally for the extensionality axiom in interesting languages, since $\forall z\,(a \in z \leftrightarrow b \in z)$ generally represents the relation of equality or indiscernibility[6]. Note that $a \mathrel{\dot\subset} b$ is cut-absorbing when the language contains ⌧, and that *Ext* is cut-absorbing in any case.

It is easy to see that if, for every $\mathcal{M}$ and v, $\mathcal{M}, v \models^- \psi$ entails $\mathcal{M}, v \models^- \varphi$, and ψ is cut-absorbing, then φ is cut-absorbing as well.

But we don't have in general that if, for every $\mathcal{M}$ and v, $\mathcal{M}, v \not\models^+ \psi$ entails $\mathcal{M}, v \models^- \varphi$, and ψ is cut-absorbing, then φ is cut-absorbing[7]. However, our last proposition shows that this works when ψ is *Ext*.

PROPOSITION 9. — *If a sequent $\varphi \Vdash$ Ext is cut-free derivable, then φ is cut-absorbing.*

PROOF. — By invertibility of the rules $\forall_R$ and $\to_R$ (proposition 3), we can suppose that a cut-free derivation of $\varphi \Vdash$ *Ext* ends in something of the form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\varphi, a \sim b, a \in d \Vdash b \in d}}{\varphi, a \sim b \Vdash (a \in d \to b \in d)}\,{\to_R}}{\varphi, a \sim b \Vdash a \mathrel{\dot\subset} b}\,\forall_R}{\dfrac{\varphi \Vdash (a \sim b \to a \mathrel{\dot\subset} b)}{\varphi \Vdash \mathit{Ext}}\,\forall_R}\,{\to_R}$$

where the a, b, d are distinct proper constants.

If we remove from the prederivation above $\varphi, a \sim b, a \in d \Vdash b \in d$ the formulas $a \in d$ and $b \in d$ that are not ancestors of φ—remember that the sequents are pairs of sets, not of sequences or of multisets!—, we remain with a prederivation, ending in $\varphi, a \sim b \Vdash$, in which a, b, d do no longer act as proper constants. Let's replace a and b everywhere by a fresh constant c. The result is a cd-derivation of $\varphi, c \sim c \Vdash$.

We will conclude that $\varphi \Vdash$ is glut derivable from the following three very simple cases of cut-elimination.

(1) If $\Gamma \Vdash s \in c, \Delta$ and $\Gamma, s \in c \Vdash \Delta$ are cd-derivable, then $\Gamma \Vdash \Delta$ is cd-derivable.

This we prove by induction on the length of a cd-derivation of $\Gamma, s \in c \Vdash \Delta$.

---

6. See [CRA 05]. See also [HIN 94] for a discussion of various notions of extensionality in glut logic.

7. Here are some counter-examples. If φ is cut-absorbing and $\Vdash \varphi$ cut-free derivable, $\psi \Vdash \varphi$ is cut-free derivable, for any ψ, cut-absorbing or not; $\psi \Vdash \varphi \to (\varphi \wedge \psi)$ is cut-free derivable and $\varphi \to (\varphi \wedge \psi)$ is cut-absorbing if φ is both cut-free and cut-absorbing. The same holds for $\forall x\,\psi(x) \Vdash \exists x\,\varphi(x) \to \exists x\,(\varphi(x) \wedge \psi(x))$, if $\exists x\,\varphi(x)$ is cut-absorbing and $\varphi(t)$ is cut-proof for some t.

Suppose $\Gamma, s \in c \Vdash \Delta$ is an identity or a cd-sequent, but $\Gamma \Vdash \Delta$ is not. Then $s \in c$ is in Δ and $\Gamma \Vdash \Delta$ is derivable, by hypothesis. The rest is straightforward, as $s \in c$ can never be introduced by a rule.

(2) If $\Gamma, (s \in c \leftrightarrow s \in c) \Vdash \Delta$ is cd-derivable, then $\Gamma \Vdash \Delta$ is cd-derivable.

This is again proved by induction on the length of a cd-derivation. Suppose $\Gamma, (s \in c \leftrightarrow s \in c) \Vdash \Delta$ is an identity or a cd-sequent, but $\Gamma \Vdash \Delta$ is not. Then $(s \in c \leftrightarrow s \in c)$ is in Δ, and $\Gamma \Vdash \Delta$ is cd-derivable:

$$\frac{\Gamma, s \in c \Vdash s \in c, \Delta \qquad \Gamma, s \in c \Vdash s \in c, \Delta}{\Gamma \Vdash \Delta} \leftrightarrow_R$$

Suppose $(s \in c \leftrightarrow s \in c)$ is introduced by a $\leftrightarrow_L$-rule:

$$\frac{\Gamma^* \Vdash s \in c, \Delta \qquad \Gamma^*, s \in c \Vdash \Delta}{\Gamma^*, (s \in c \leftrightarrow s \in c) \Vdash \Delta} \leftrightarrow_L$$

where $\Gamma^*, (s \in c \leftrightarrow s \in c)$ is $\Gamma, (s \in c \leftrightarrow s \in c)$. We can suppose that $(s \in c \leftrightarrow s \in c)$ is not in Γ. If Γ^* is Γ, then the result follows from (1). If Γ^* is $\Gamma, (s \in c \leftrightarrow s \in c)$, then, applying the inductive hypothesis, we get cd-derivations of $\Gamma \Vdash s \in c, \Delta$ and of $\Gamma, s \in c \Vdash \Delta$, from which we get a cd-derivation of $\Gamma \Vdash \Delta$, by (1).

In a very similar way, we prove by induction, with the help of (2), that

(3) If $\Gamma, c \sim c \Vdash \Delta$ is cd-derivable, then so is $\Gamma \Vdash \Delta$.

Therefore φ is cut-absorbing, by theorem 4. ■

Assuming that compactness can be established by generalizing the proof of theorem 4, we remark that proposition 9 shows the extreme fragility of *Ext* in the sense that if *Ext* is true in a non-classical model $\mathcal{M}$, then there exists another model in which *Ext* is NOT TRUE, while the non-false and non-true sentences in $\mathcal{M}$ remain unchanged.

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