THE JOURNAL OF SYMBOLIC LOGIC
Volume 47, Number 1, March 1982

ON THE CONSISTENCY OF AN IMPREDICATIVE SUBSYSTEM OF QUINE'S NF

MARCEL CRABBÉ

Introduction. NFP is the *predicative* fragment of NF. In this system we do not allow a set to exist if it cannot be defined without using quantifiers ranging over its type or parameters of a higher type. NFI is a less restrictive fragment located between NFP and NF.

We show that NFP is really weaker than NFI; similarly, NFI is weaker than NF. This result will be obtained in the following manner: on the one hand, we will show that NFP can be proved consistent in elementary arithmetic and that second order arithmetic is interpretable in NFI; on the other hand, we will prove the consistency of NFI in third order arithmetic, which is contained in NF.[1]

The paper is divided in four sections. In §1, we define the concepts needed and collect a few results together in such a way that they will be ready for later use. In §2, we will present a model-theoretic (quick) proof of the consistency of NFI (and thus of NFP). The proof will be chosen (it is not the quickest!) so as to motivate in a natural manner the details of the proof-theoretical version of it that will be presented in §3. §4 will be devoted to the axiom of infinity in NFP and NFI.

§1. Definitions, notations and preliminary results.

1. NF is the theory in the language of ZF whose axioms are extensionality and the instances of the comprehension schema: $\exists y\, \forall x(x \in y \leftrightarrow \varphi)$, where φ is a stratifiable formula and y is not free in φ. NFP and NFI are subsystems of NF obtained by restricting the comprehension axioms. In NFP there must be a stratification such that the indices associated to the bound variables in φ do not exceed the type of x and the indices of the free variables in φ do not exceed the type of y. In NFI the indices associated to the variables *bound or free* in φ do not exceed the index attributed to y. So NFP is a part of NFI.

For any stratifiable φ, we can prove in NFP that $\exists y \forall z(z \in y \leftrightarrow \exists x(z = \{\dots\{x\}\dots\} \wedge \varphi))$, provided the singleton operation is sufficiently iterated. Therefore, if U is the axiom of union: $\exists y \forall x\,(x \in y \leftrightarrow \exists v(v \in z \wedge x \in v))$ we obtain

LEMMA 1.[2] NFP + U = NFI + U = NF.

If $n \geq 1$, NFP_n and NFI_n are the fragments of NFP and NFI, respectively,

Received December 10, 1979.

[1]An introduction to NF and type theories can be found in [11] and [7]. Some useful and recent results about these systems are presented and much simplified in [2].

[2]U can be restricted to the assertion that the union of a set of unit sets always exists : $\forall x(x \subseteq \text{USC}(V) \to \exists y(x = \text{USC}(y))$.

0022-4812/82/4701-0012/$01.60

obtained by keeping as comprehension axioms only those that are stratifiable with the first n indices: $1, \ldots, n$. E' is the axiom of NFP_4 asserting the existence of the set of sets whose intersection is not empty: $\exists y \forall x(x \in y \leftrightarrow \exists t \forall z(z \in x \rightarrow t \in z))$.

The next theorem is essentially due to Grishin [9] and can be checked easily from his proof:

FIRST REDUCTION THEOREM.[3] $\mathrm{NFP} = \mathrm{NFP}_3 + E'$; $\mathrm{NFI} = \mathrm{NFI}_3 + E'$.

2. Let TP and TI be the theories of types corresponding to NFP and NFI, respectively.[4] In the following, T will be either TP or TI. If $n \geq 1$, T_n is the fragment of T built on the first n types: $1, \ldots, n$. T_n^∞ is the theory of the infinite models of T_n, that is, T_n plus, for each m, the standard axiom expressing that there are at least m objects of type 1.

If φ is a formula of the language of T, then φ^+ will be obtained from φ by raising all type indices by 1. $T_4^{(3)}$ is the fragment of T_4 whose comprehension axioms are those of T_3 and those of the form φ^+ whenever φ is an axiom of T_3. E is the axiom of TP_4 resulting from E' by putting the type indices 1, 2, 3 and 4 at the right places. Amb (for "ambiguity") is the set of all the sentences (i.e. closed formulas) of the form $\varphi \leftrightarrow \varphi^+$ formulated in the language of T_4 (thus φ is in the language of T_3).

SECOND REDUCTION THEOREM. NFP (*resp.* NFI) *is consistent iff* $TP_4^{(3)}$ (*resp.* $TI_4^{(3)}$) $+ E +$ Amb *is consistent.*

The proof follows from [13] and the first reduction theorem.

3. A model of a set of sentences of the language of T_n is a structure $\langle M_1, \ldots, M_n; <_1, \ldots, <_{n-1}\rangle$ satisfying these sentences (in the appropriate sense), where the M_i's $(1 \leq i \leq n)$ are pairwise disjoint sets and for each i $(1 \leq i \leq n-1)$, $<_i$ is a relation included in the cartesian product $M_i \times M_{i+1}$.

A model $\langle M_1, M_2, <_1\rangle$ of T_2 is called *countably saturated* if:

M_2 is countably infinite,

for every object a of M_2 such that $\{x \in M_1 | x <_1 a\}$ is infinite there is a b in M_2 such that $\{x \in M_1 | \ x <_1 b$ and $x <_1 a\}$ and $\{x \in M_2 | \ x \nless_1 b$ and $x <_1 a\}$ are both infinite.

LEMMA 2. (1) *Two countably saturated models of* T_2 *are isomorphic.*

(2) *Every countable model* $\mathcal{M}$ $(\mathcal{M} = \langle M_1, \ldots, M_n, <_1, \ldots, <_{n-1}\rangle)$ *of* T_n *has an elementary extension such that, for each* i $(1 \leq i \leq n-1)$, $\langle M_i, M_{i+1}, <_i\rangle$ *is countably saturated.*

The first part is proved in [8]. The second is obtained by taking a recursively saturated extension of $\mathcal{M}$ (see [10] for refinements).

§2. The model-theoretical consistency proof. Let $\varphi_1, \ldots, \varphi_r$ $(r > 0)$ be a sequence of sentences of the language of T_3. We consider the 2^r conjunctions $\varphi_1^{\varepsilon_1} \wedge \cdots \wedge \varphi_r^{\varepsilon_r}$ where $\varphi_i^{\varepsilon_i}$ is φ_i or $\neg\varphi_i$ according to whether ε_i is 0 or 1. Let $\psi_1, \ldots, \psi_k$ be *all* the conjunctions of this sort that are consistent with T_3^∞. $\mathrm{Amb}(\varphi_1, \ldots, \varphi_r)$ is the sentence $(\varphi_1 \leftrightarrow \varphi_1^+) \wedge \cdots \wedge (\varphi_r \leftrightarrow \varphi_r^+)$. We show that $T_4^{(3)} + E + \mathrm{Amb}(\varphi_1, \ldots, \varphi_r)$ is consistent. From this and the second reduction theorem we may then conclude by compactness that NFP and NFI are consistent.

[3]For alternative reductions see [1] and [5].

[4]These systems are studied in [5].

From now on we fix r, k, $\varphi_1, \ldots, \varphi_r, \psi_1, \ldots, \psi_k$ as described. A simple application of Lemma 2 gives

LEMMA 3. *If $\langle M_2, M_3, <_2\rangle$ is a countably saturated model of T_2, then for each i $(1 \leq i \leq k)$ there is a set M_{1i} and a relation $<_{1i}$ such that $\langle M_{1i}, M_2, M_3, <_{1i}, <_2\rangle$ is a model of $T_3 + \psi_i$.* $\square$

Let $\langle M_2, M_3, <_2\rangle$ be a countably saturated model of T_2 and choose $M_{11}, \ldots, M_{1k}, <_{11}, \ldots, <_{1k}$ as indicated in Lemma 3. If $1 \leq i \leq k$, let E_i be the set of the objects a of M_3 such that

$$\langle M_{1i}, M_2, M_3, <_{1i}, <_2\rangle \models \exists z^1 \forall v^2 (v^2 \in a \rightarrow z^1 \in v^2).$$

We then have

LEMMA 4. *There is a model $\langle M_2, M_3, M_4, <_2, <_3\rangle$ of T_3 such that:*
M_4 is a collection of subsets of M_3 and $<_3$ is the standard $\in$-relation;
each E_i $(1 \leq i \leq k)$ belongs to M_4.

PROOF. M_4 may be taken as the power set of M_3, i.e. the collection of *all* subsets of M_3.

This works in the two cases. But, when T_3 is the predicative theory one can proceed in a more economical way, since M_4 may then be taken as the collection of the subsets *definable* from parameters belonging to $M_2 \cup M_3 \cup \{E_1, \ldots, E_k\}$.

LEMMA 5. *$T_4^{(3)} + E + \mathrm{Amb}(\varphi_1, \ldots, \varphi_r)$ is consistent.*

PROOF. Let $\mathscr{M} = \langle M_2, M_3, M_4, <_2, <_3\rangle$ be as in Lemma 4. There is exactly one ε such that $\varphi_r^{\varepsilon_1} \wedge \ldots \wedge \varphi_r^{\varepsilon_r}$ is true in $\mathscr{M}$. But since $\mathscr{M}$ is a model of T_3^∞ there is a unique i $(1 \leq i \leq k)$ such that this sentence is ψ_i. So, the structure $\mathscr{M}'$ $(\mathscr{M}' = \langle M_{1i}, M_2, M_3, M_4, <_{1i}, <_2, <_3\rangle)$ is a model of $T_4^{(3)}$ satisfying ψ_i and ψ_i^+. Moreover, E_i belongs to M_4. $\mathscr{M}'$ is thus a model of $T_4^{(3)} + E + \mathrm{Amb}(\varphi_1, \ldots, \varphi_r)$.

§3. The proof-theoretical version. We show here how to transform the consistency proof given above into one which uses proof-theoretical means only. Besides its own interest, such a demonstration will provide information about the relative powers of the considered systems.

1. *More definitions.* If i is a natural number and φ a formula of T_n, then φ^i will be the expression resulting from φ by replacing in it the type index 1 by $1i$ and $T_n^{\infty i}$ will be the theory differing inessentially from T_n^∞ in that the type 1 is denoted in it by $1i$. T_n^* is the union of the theories $T_n^{\infty i}$ in the sense that the nonlogical axioms of T_n^* are just those φ^i's that are nonlogical axioms of the $T_n^{\infty i}$'s.[5] In this section, n will always be 3 or 4, and r, k, $\varphi_1, \ldots, \varphi_r, \psi_1, \ldots, \psi_k$ are fixed as in §2. Ψ is $\psi_1^1 \wedge \cdots \wedge \psi_k^k$, which is a sentence in the language of T_3^*. Note that the consistency of T_3^∞ and that $k \geq 1$ are elementary provable.

2. LEMMA 6. *It is elementary provable, from the definition of Ψ, that $T_3^* + \Psi$ is consistent.*

PROOF. Using the elementary proof of the quantifier elimination theorem for the theory of atomic boolean algebras (extended by some extra predicates), one

[5]The type 1 is thus split into k parts. More precisely, the language of T_n^* has variables for the "types" $11, \ldots, 1k, 2, \ldots, n$; equality symbols : $=_{11}, \ldots, =_{1k}, =_2, \ldots, =_n$; relation symbols : $\in_{11}, \ldots, \in_{1k}, \in_2, \ldots, \in_{n-1}$. Formulas are built as usual from atomic ones of the kind : $x^{1i} =_{1i} y^{1i}$, $x^{1i} \in_{1i} y^2$, $x^m \in_m y^{m+1}$, etc.

proves in arithmetic that $TP_2^\infty = TI_2^\infty$ is a complete theory [3], [5]. On the other hand, Robinson's consistency theorem is elementary provable [15]. These proofs can be transfered to the typed theories.

LEMMA 7. *T_4^* is a conservative extension of T_3^*. This is provable in first order arithmetic in the predicative case and in third order arithmetic when T_4^* is TI_4^*.*

PROOF. T_4^* can be presented in Gentzen's style as a "second order" sequent calculus. If Γ and Δ are finite sequences of formulas of the language of T_3^*, we let $\Gamma \Vdash_1 \Delta$ mean that the sequent $\Gamma \Vdash \Delta$ is provable in the sequent calculus (without comprehension or equality rules) adapted to the language of T_3^*. We introduce then a metalinguistic abstract $\{x^3 \mid \varphi\}$ for each formula φ of T_4^*, if T_4^* is TI_4^*; and for each φ containing no bound variables of type 4 if T_4^* is TP_4^*. We let $\Gamma \Vdash_2 \Delta$ mean that $\Gamma \Vdash \Delta$ is provable in the corresponding "second order" calculus (without equality or extensionality rules).

From the cut elimination theorem for the predicative second order logic (which is provable in first order arithmetic [15]) and the cut elimination for full (impredicative) second order logic (provable in third order arithmetic [14]), it is clear that if the formulas of Γ and Δ are in T_3^*, then $\Gamma \Vdash_2 \Delta$ entails $\Gamma \Vdash_1 \Delta$.

Let Eq_1 be the axiom

$$\forall x^3 \forall y^3 (x^3 = y^3 \to \forall z^4 (x^3 \in z^4 \to y^3 \in z^4))$$

and Eq_2 the conjunction of the axioms for equality and extensionality for 4-typed objects. If φ is a formula of T_3^*, then $T_4^* \vdash \varphi$ implies that Γ, Eq_1, $\mathrm{Eq}_2 \Vdash_2 \varphi$ for a finite sequence Γ of axioms of T_3^*. If we replace, in a proof of such a sequent, $x^4 = y^4$ by $\forall z^3(z^3 \in x^4 \leftrightarrow z^3 \in y^4)$, and if we then relativise the x^4's to the unary relation $R(x^4)$ defined by

$$\forall x^3 \forall y^3 (x^3 = y^3 \wedge x^3 \in x^4 \to y^3 \in x^4),$$

we obtain that $\Gamma \Vdash_2 \varphi$ (see [15] for details). But then $\Gamma \Vdash_1 \varphi$, that is, $T_3^* \vdash \varphi$.[6]

LEMMA 8. *$T_4 + \mathrm{Amb}(\varphi_1, \ldots, \varphi_r)$ is consistent.*

PROOF. A will abreviate $\mathrm{Amb}(\varphi_1, \ldots, \varphi_r)$. First, for every i $(1 \le i \le k)$, $T_4^* + \Psi \vdash \psi_i^+ \to A^i$. But, if $T_4 \vdash \neg A$, then $T_4^* \vdash \neg A^i$, for every i $(1 \le i \le k)$. Thus, if the lemma is not true, it follows that $T_4^* + \Psi \vdash \neg \psi_1^+ \wedge \cdots \wedge \neg \psi_k^+$. By the definition of the sequence $\psi_1, \ldots, \psi_k$, we have that $T_3^\infty \vdash \psi_1 \vee \cdots \vee \psi_k$. Hence, $T_4^* \vdash \psi_1^+ \vee \cdots \vee \psi_k^+$. So, $T_4^* + \Psi$ would be inconsistent. This is clearly impossible in view of Lemmas 6 and 7. $\square$

If m is a positive natural number, we let PA_m be the system of mth order arithmetic (PA is PA_1). $\mathrm{CON}(S)$ is the canonical sentence of PA expressing the consistency of S.

THEOREM 1. *$PA \vdash \mathrm{CON}(\mathrm{NFP})$ and $PA_3 \vdash \mathrm{CON}(\mathrm{NFI})$.*

PROOF. Lemmas 6, 7 and 8 show that $PA \vdash \mathrm{CON}(TP_4 + \mathrm{Amb})$ and PA_3

[6] This proof can be adapted in order to give a proof-theoretical proof of the fact that ML (Quine's Mathematical Logic) is a conservative extension of NF.

$\vdash$ CON(TI_4 + Amb). $PA \vdash$ CON(TP_4 + Amb) $\rightarrow$ CON(NFP_4) and $PA \vdash$ CON(TI_4 + Amb) $\rightarrow$ CON(NFI_4) follow from [4] and [6], where a finitary proof of Specker's result can be found. $PA \vdash$ CON(NFP_4) $\rightarrow$ CON(NFP) and $PA \vdash$ CON (NFI_4) $\rightarrow$ CON(NFI) is clear from [9].

§4. The axiom of infinity. In NFP, we can define the notion of natural number as usual:

$$Nn(x) \equiv_{\text{def}} \forall y(\text{Ind}(y) \rightarrow x \in y),$$

where

$$\text{Ind}(y) \equiv_{\text{def}} \{\Lambda\} \in y \wedge \forall x(x \in y \rightarrow x + 1 \in y)$$

and

$$x + 1 \equiv_{\text{def}} \{z \mid \exists t\, (t \in z \wedge z \backslash \{t\} \in x\}.$$

The axiom of infinity says that V, the set of all sets, is not an element of a natural number or, alternatively, that Λ, the empty set, is not a natural number.

THEOREM 2. NFP $\vdash \neg\, Nn(\Lambda)$; NFP $\vdash \forall x\, (Nn(x) \rightarrow V \notin x)$.

PROOF. If the axiom of union holds, then, by Lemma 1, Specker's famous proof of the axiom of infinity for NF [12] goes through.[7]

Suppose now that U is false and let X be $\{x \mid \Lambda \neq x \wedge \forall y(y \in x \rightarrow \forall z(z \subseteq y \rightarrow Uz \text{ exists}))\}$. X is predicatively defined when "Uz exists" abbreviates $\exists y \forall x(x \in y \leftrightarrow \exists v(v \in z \wedge x \in v))$. Note that $\neg U$ is equivalent to $\neg \forall z(z \subseteq V \rightarrow Uz$ exists). The theorem will be proved if we can establish that NFP $\vdash$ Ind(X).

It is easy to see that $\{\Lambda\} \in X$. Assume then that $x \in X$. This implies $x \neq \Lambda$ and $V \notin x$ (if not, U would hold). Also, $x + 1 \neq \Lambda$ because if $y' \in x$ and $t \notin y'$, then $y' \cup \{t\} \in x + 1$. Let y, z, t be such that $y \in x + 1$, $z \subseteq y$, $t \in y$ and $y\backslash\{t\} \in x$. It remains to show that Uz exists. Two cases are possible. (1) If $t \notin z$, $z \subseteq y\backslash\{t\}$, $y\backslash\{t\} \in x$, and Uz exists. (2) If $t \in z$, $z\backslash\{t\} \subseteq y\backslash\{t\}$, $y\backslash\{t\} \in x$, $U(z\backslash\{t\})$ exists and, because finite union is predicatively definable, $U(z\backslash\{t\}) \cup t = Uz$ exists. □

Although the axiom of infinity is a theorem of NFP, it cannot be proved in this system that the *set* Nn ($Nn = \{x \mid Nn(x)\}$) of natural numbers exists. This would in fact entail that PA could be interpreted in NFP and thus that the consistency of NFP could be proved in NFP.

Despite this, the axiom of infinity, being provable, ensures that nontrivial weak fragments of arithmetic are interpretable in NFP.

The situation is much different for NFI, since there the set of natural numbers exists. This fact permits us to interpret not only PA but also classical analysis. If we remark, moreover, that PA_3 is interpretable in NF, then, from Theorem 1 follows

THEOREM 3. NFI $\vdash$ CON(NFP) *and* NF $\vdash$ CON(NFI).

[7] Specker's argument needs U, in the restricted form of footnote 2, for showing that USC(SC(x)) and SC(USC(x)) are equipollent.

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UNIVERSITÉ DE LOUVAIN (U.C.L.)
INSTITUT SUPÉRIEUR DE PHILOSOPHIE
LOUVAIN-LA-NEUVE, BELGIUM