Ambiguous Cardinals*

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Abstract

A cardinal number κ is said to be "ambiguous" if it is indiscernible from 2 to the power of κ. In a more specific way, κ is ambiguous if the natural typed structure over a set X of size κ is elementarily equivalent to the natural structure over the power set of X. Some striking results arising from the method Specker used to refute AC in NF will be extracted from the NF context and used to give results in the more usual ZF theory.

1 Strong ambiguity

The fact that no set can be equal to the set of its subsets contradicts our primitive intuition that there is nothing more dreamt of in our universe of sets, than things in heaven and earth, i.e. that $\mathcal{P}(U)$ is not greater than U.

Among the possible different ways out of this predicament, I will consider only the following one: while not insisting that $\mathcal{P}(U) = U$, one should at least expect that U and $\mathcal{P}(U)$ could be somehow indiscernible.

Taken literally, even this is not true, since one can define a notion of height of a set: $h(X)$ is the cardinal of the set $\{..., X_2, X_1, X\}$, where $\mathcal{P}(X_{i+1}) = X_i$. Now, since this set is finite[1], $h(X)$ is even iff $h(\mathcal{P}(X))$ is odd.

Actually, the way Cantor showed that $X \neq \mathcal{P}(X)$ was by showing that $|X| \neq |\mathcal{P}(X)|$. It seems therefore natural to limit the indiscernibility requirement to properties of cardinals.

The consistency of ZF plus the existence of a cardinal κ indiscernible from 2^κ, in an unqualified sense, is open. The same question with ZFC is easily settled as we now show.

*This writing is a sequel to Maurice Boffa's last paper [1].

[1] Use Sierpiński's theorem (see below) or simply the fact that in ZF the rank of $\mathcal{P}(X)$ is the rank of X plus one.

Before we proceed, let us point out that, since the axiom of choice will not be assumed, cardinals will be implemented in ZF as equivalence classes of lowest rank of the equipollence relation.

Definitions

$\Phi(\alpha)$ *is the set* $\{\alpha, 2^\alpha, ...\}$;

$\Phi_\mu(\alpha)$ *is the set* $\Phi(\alpha) \cap \{\, \beta \mid \beta \leq \mu \,\}$.

Thus, $\Phi_\mu(\alpha)$ is infinite iff $\Phi_\mu(\alpha) = \Phi(\alpha)$.

A well-ordered cardinal *is the cardinal of a well-ordered set.*

Proposition 1 *If μ is well-ordered, then there is a property discriminating μ from 2^μ.*

Proof

If α and β are cardinals, let $\alpha \sim \beta$ mean that $\Phi(\alpha) \cap \Phi(\beta)$ is not empty. One checks first that $\sim$ is an equivalence relation[2]. One then defines a notion of distance between cardinals $\alpha \sim \beta$, by taking the cardinal of $(\Phi(\alpha) \cup \Phi(\beta)) \setminus (\Phi(\alpha) \cap \Phi(\beta))$. Assuming μ well-ordered, let 0^* be the smallest well-ordered cardinal $\sim$-related with μ. It is evidently also the smallest well-ordered cardinal which is $\sim$-related with 2^μ, because $\mu \sim 2^\mu$. Therefore, the distances from 0^* to μ and from 0^* to 2^μ are not the same parity. Thus the property "the distance from 0^* to the cardinal of x is even" discriminates between μ and 2^μ. ∎

2 Typical ambiguity and AC

If we are to look for weaker forms of ambiguity, the obvious candidate to consider for this is the expression in ZF of the kind of ambiguity arising from the very simple type theory related to NF, that was expounded in [7].

Definitions

A typed formula *is a formula of the language of (simple) type theory,* TT, *i.e. built up in the usual way from atomic formulas of the form $x^i \in y^{i+1}$ or $x^i = y^i$.*

$\langle\langle x \rangle\rangle$ *denotes the structure $\langle x, \mathcal{P}(x), ...; \in \rangle$, which is a natural model of* TT *(as long as x is not empty!).*

[2]This is an instance of the general fact that if f is a function, then the relation that obtains between x and y iff $\exists k \exists q\, f^k(x) = f^q(y)$ is an equivalence relation.

The following trivial reflection property clearly holds: if the sentence A is typed with $0, 1, ..., k$, then $\langle\langle x \rangle\rangle \models A$ iff $\langle x, ..., \mathcal{P}^k(x); \in\rangle \models A$.

Lemma 1 *If $|K| = |M|$, then $\langle\langle K \rangle\rangle \models A$ iff $\langle\langle M \rangle\rangle \models A$.*

PROOF

Let f be a bijection between K and M and k the maximum type in A. One defines a sequence of $k+1$ bijections by letting $f_0 = f$ and, for $0 < n < k$, $f_{n+1}(x) = \{ f_n(y) \mid y \in x \}$. The sequence $\langle f_0, ..., f_k\rangle$ is an isomorphism between the structures $\langle K, ..., \mathcal{P}^k(K); \in\rangle$ and $\langle M, ..., \mathcal{P}^k(M); \in\rangle$.

$$\begin{array}{ccc} \mathcal{P}^k(K) & \xrightarrow{f_k} & \mathcal{P}^k(M) \\ & \vdots & \\ \mathcal{P}(K) & \xrightarrow{f_1} & \mathcal{P}(M) \\ K & \xrightarrow{f_0} & M \end{array}$$

■

Definitions

Lemma 1 says that, given a typed sentence A, whether or not $\langle\langle x \rangle\rangle \models A$, is determined solely by the size of x. This suggests the following definition:
$\mu \models A$ iff $\forall x\, (\mu = |x| \to \langle\langle x \rangle\rangle \models A)$.
Thus, $|x| \models A$ iff $\langle\langle x \rangle\rangle \models A$.

Following Boffa, a formula $P(x)$ will be called a typed property[3] *iff there is a typed sentence A such that $\forall x\, (P(x) \leftrightarrow x \models A)$ holds in* ZF.

Cardinal μ is indiscernible *from cardinal κ, which we notate $\mu \equiv \kappa$, if and only if $\mu \models A$ iff $\kappa \models A$, for every typed sentence A. Thus, indiscernible cardinals are cardinals with the same typed properties.*

μ is ambiguous *iff μ is indiscernible from 2^μ.*

[3]Strictly speaking, our definition differs from Boffa's one in that it takes advantage of lemma 1 and only makes sense for non-zero cardinal numbers.

Notice that $\mu \equiv \kappa$ is not expressible in the language of TT*, but can be written in a single formula in* ZF*, using truth definitions or Fraïssé's n-isomorphisms.*

Let us recall that Hartogs' *cardinal* number $\aleph(\mu)$ *is the least well-ordered cardinal not less or equal to μ. It is known from a result by Sierpiński that $\aleph(\mu) < 2^{2^{2^{\mu}}}$.*

Although some sophistication is required to determine whether or not a property of cardinals is typed, it is useful to bear in mind that, roughly speaking, a typed property is one that can be checked by considering a proper segment of the typed structure over a set of the size of the argument. In doing this, it might be helpful to exploit the USC *and the associated* T *operations, in order to be able to compare sizes at different type levels. For example, to show that $\aleph(x) \leq 2^x$ is a typed property, one has to find a typed sentence equivalent to the assertion that the Hartogs' number of the cardinal $\nu = |\{ x^0 \mid x^0 = x^0 \}|$ of the universe in type 1 is less than the cardinal of the universe in type 2. This can be done by saying, in the typed language, that there is a well-ordered cardinal less or equal to $\nu^+ = |\{ x^1 \mid x^1 = x^1 \}|$, but not less than $\mathsf{T}(\nu)$.*

Another paradigmatic example is "$\exists z\,(z < x \wedge \Phi_x(z)$ is finite $)$". To see that this is a typed property, it suffices to notice that: $\exists z^2 \exists y^3\,(z^2 < \nu \wedge y^3$ is finite $\wedge$ $z^2 \in y^3 \wedge \forall v^1\,(|v^1| \in y^3 \rightarrow \mathsf{T}^{-1}(|\mathcal{P}(v^1)|) \in y^3))$ abbreviates a typed sentence[4].

On the other hand, the property mentioned in proposition 1 is at first sight not typed. As a matter of fact, it was shown in [2] that it cannot *be so.*

Though we will not treat this in this paper, it is worth noticing at this stage that a stronger —perhaps more appealing in the ZF-context— notion of ambiguity than typical ambiguity can be introduced by considering the cumulative hierarchy, up to a given ordinal, over a non-empty set, that is viewed as a set of atoms. This is effected by taking a set of the same size containing only non empty sets, and then emptying them:

$$\begin{aligned} y \in_x z &\equiv y \in z \wedge z \notin \mathsf{USC}(x) \\ \mathcal{P}_x(y) &= \mathcal{P}(y) \setminus \mathsf{USC}(x) \\ x_0 &= \mathsf{USC}(x) \\ x_{\alpha+1} &= x_\alpha \cup \mathcal{P}_x(x_\alpha) \\ x_\lambda &= \bigcup_{\alpha<\lambda} x_\alpha \end{aligned}$$

The analogue of lemma 1 holds for $\langle x_\alpha, \in_x \rangle \models A$, when A is a ZF-sentence. Consequently, we may meaningfully write, for example, $\mu_\alpha \models A$ for $\forall y\,(|y| = \mu \rightarrow \langle y_\alpha, \in_y$

[4]It is assumed as usual that $T^{-1}(T(\alpha)) = \alpha$ and $T^{-1}(x) = \emptyset$, if x is not of the form $T(\alpha)$.

$\rangle \models A$), which is a formula with one variable free in case α is definable, and introduce notions of indiscernibility and ambiguity, in the same manner as above. Thus the property used in proposition 1 although not typed, is of the kind $x_{\omega+1} \models A$.[5]

Proposition 2 *If μ is ambiguous, so is 2^μ.*

Proof

Using the type raising operation $...^+$, we have $2^\mu \models A$ iff $\mu \models A^+$ iff $2^\mu \models A^+$ iff $2^{2^\mu} \models A$. ■

Proposition 3 *If μ is ambiguous, then $\aleph(\mu) \leq 2^\mu$.*

Proof

Assuming μ ambiguous, 2^μ is also ambiguous, by proposition 2, and thence $\mu \equiv 2^{2^\mu}$. Suppose now that $\aleph(\mu) \not\leq 2^\mu$, which is $\aleph(\mu) = \aleph(2^\mu)$. Then, since the property of cardinals $\aleph(x) = \aleph(2^x)$ is typed, we have $\aleph(\mu) = \aleph(2^{2^{2^\mu}})$, contradicting $\aleph(\mu) < 2^{2^{2^\mu}}$. ■

Corollary 1 *If $\Phi_\mu(\delta)$ is infinite, so is $\Phi_\mu(\aleph(\delta))$.*

Proof

Every element in $\Phi_\mu(\aleph(\delta))$ is less than an element in $\Phi(\delta)$, since $\aleph(\delta) < 2^{2^{2^\delta}}$. Hence if $\Phi_\mu(\delta)$ is infinite, $\Phi_\mu(\delta) = \Phi(\delta)$, and $\Phi_\mu(\aleph(\delta))$ is also infinite. ■

Definitions

$\delta \in \mathsf{WG}(\mu)$ (well-generated from μ) iff there is a well-ordered cardinal $\gamma \leq \mu$ such that $\delta \in \Phi_\mu(\gamma)$.

0_μ is the least well-ordered cardinal δ such that $\Phi_\mu(\delta)$ is finite. Observing that $\Phi_\mu(\aleph(\mu))$ is empty, we see that 0_μ always exists. Moreover, $\Phi_\mu(0_\mu)$ is not empty iff $0_\mu \leq \mu$.

The rest of this section combines arguments stemming from Specker [6], and expanded by Pétry [5], by means of which we aim to express in ZF the substantial elements of Specker's original proof.

Lemma 2

If μ is ambiguous and $0_\mu \leq \mu$, then $\Phi_{2^\mu}(0_\mu)$ is infinite.

[5] To make things work nicely, we have to suppose that in $\langle x_\alpha, \in_x \rangle$, the power set of a set y is the set of all sub*sets* of y, namely $\mathcal{P}_x(y)$. It is of course possible that $\mathcal{P}_x(y) \notin x_\alpha$, even when $y \in x_\alpha$.

Proof

Suppose μ ambiguous, $0_\mu \le \mu$, but $\Phi_{2^\mu}(0_\mu)$ finite. Then $0_{2^\mu} = 0_\mu$. Indeed, $0_{2^\mu} \le 0_\mu$, as $\Phi_{2^\mu}(0_\mu)$ is finite. Now, $\Phi_\mu(0_{2^\mu})$ is finite as well, because $\Phi_\mu(0_{2^\mu}) \subseteq \Phi_{2^\mu}(0_{2^\mu})$. Hence $0_\mu \le 0_{2^\mu}$.

If δ is the last element in $\Phi_\mu(0_\mu)$, then 2^δ belongs to $\Phi_{2^\mu}(0_\mu) = \Phi_{2^\mu}(0_{2^\mu})$. Therefore, $k_\mu = |\Phi_{2^\mu}(0_\mu) \setminus \Phi_\mu(0_\mu)|$ is a non-zero natural number.

There is a typed property $P(x)$ saying that the quotient of the division of $|\Phi_x(0_x)|$ by k_x is even[6] — i.e. that the cardinal of $\Phi_x(0_x)$ is of the form $k_x \cdot 2q + r$, for some $r < k_x$. Clearly, one has $P(\mu)$ iff $\neg P(2^\mu)$, contrary to the fact that μ is ambiguous. ∎

Lemma 3 *If μ is ambiguous and $0_\mu \not\le \mu$, then $\Phi_{2^\mu}(\aleph(\mu))$ is infinite.*

Proof

Observe that $0_{2^\mu} \not\le 2^\mu$, because $0_x \not\le x$ is a typed property[7]. The result then immediately follows, by proposition 3. ∎

Theorem 1 *If μ is ambiguous, then $\Phi_{2^\mu}(\delta)$ is infinite, for some $\delta \in \mathsf{WG}(2^\mu)$ such that $\delta \not\le \mu$.*

Proof

If $0_\mu \not\le \mu$, then, by lemma 3, δ can be taken as $\aleph(\mu)$.

If $0_\mu \le \mu$, then $\Phi_{2^\mu}(0_\mu)$ is infinite, by lemma 2. Now, we let δ be 2^λ, where λ is the greatest element in $\Phi_\mu(0_\mu)$, and we are done. ∎

Corollary 2 *If μ is ambiguous and $0_\mu \le \mu$, then, for some $\delta \le \mu$ in $\mathsf{WG}(\mu)$, 2^δ is not comparable with μ.*

Proof

By lemma 2, $\Phi_{2^\mu}(0_\mu)$ is infinite. We let δ be the last element in $\Phi_\mu(0_\mu)$. Then $2^\delta \not\le \mu$. And if $\mu \le 2^\delta$, then $2^\mu \le 2^{2^\delta}$ and $\Phi_{2^\mu}(2^\delta)$ has two elements at most: $\Phi_{2^\mu}(2^\delta)$ is $\{2^\delta\}$ or $\{2^\delta, 2^{2^\delta}\}$. This is impossible, as it would make $\Phi_{2^\mu}(0_\mu)$ finite. ∎

Corollary 3 *If μ is ambiguous, then there is a cardinal $\delta \le 2^\mu$ in $\mathsf{WG}(2^\mu)$, not comparable with μ.*

[6] This is analogous to a proof by Forster, that in NF, $n - Tn$ is not a cantorian positive number.

[7] It suffices to observe that $\exists y\,(y$ is a well-ordered cardinal $\wedge\, y \le x \wedge \Phi_x(y)$ is finite$)$ is typed.

Proof

If $0_\mu \leq \mu$, we apply corollary 2.

Else, we can take $\aleph(\mu)$ for δ. Indeed, if $\mu < \aleph(\mu)$, then $2^\mu \leq 2^{\aleph(\mu)}$ and $\Phi_{2^\mu}(\aleph(\mu))$ is either $\{\aleph(\mu)\}$ or $\{\aleph(\mu), 2^{\aleph(\mu)}\}$. This conflicts with lemma 3. ∎

3 Hartogs' numbers

Now we will give a condition under which $\aleph(\mu)$ can play the role of the δ mentioned in theorem 1.

Lemma 4 *Let μ be ambiguous. $2^{\aleph(\mu)}$ is well-ordered if and only if 2^δ is well-ordered, for all well-ordered $\delta \leq 2^\mu$.*

Proof

Suppose first that $2^{\aleph(\mu)}$ is well-ordered. Since "$2^{\aleph(x)}$ is well-ordered" is a typed property, this amounts to suppose that $2^{\aleph(2^\mu)}$ is well-ordered.
Then, if $\delta \leq 2^\mu$ and δ is well-ordered, we have $\delta \leq \aleph(2^\mu)$ and $2^\delta \leq 2^{\aleph(2^\mu)}$. This shows that 2^δ is well-ordered.

Conversely, $\aleph(\mu) \leq 2^\mu$, by proposition 3, and it follows, from the hypothesis, that $2^{\aleph(\mu)}$ is well-ordered. ∎

Theorem 2 *If μ is ambiguous and $2^{\aleph(\mu)}$ is well-ordered, then $\Phi_{2^\mu}(\aleph(\mu))$ is infinite.*

Proof

Suppose that μ is ambiguous and that $2^{\aleph(\mu)}$ is well-ordered. By theorem 1, there is $\delta \in \mathsf{WG}(2^\mu)$ such that $\delta \not\leq \mu$ and $\Phi_{2^\mu}(\delta)$ is infinite. Since, by lemma 4, such a δ is well-ordered, we have $\aleph(\mu) \leq \delta$. Therefore, $\Phi_{2^\mu}(\aleph(\mu))$ is infinite. ∎

4 Generalizations

This last section is devoted to generalizing theorems 1 and 2. The proofs will be omitted, as they can be obtained by scrutinizing the corresponding proofs of the original theorems. Stress will be laid instead on generalizing the concepts involved; mainly the $2^{\cdots}$ function and the notion of ambiguity.

We start by introducing a notion of *jump* that will enable us to exploit more generally those properties of the $2^{\cdots}$ function that are essential to the arguments that use

Specker's methods. This will enable not only to use a kind of general function, but also to export, to ZF, arguments from NFU, that were produced by Holmes, Forster, Boffa and myself. Indeed, in NFU, we don't necessarily have that the universe U is the same size as its power set $\mathcal{P}(U)$, though we know that $\mathcal{P}(U) \subseteq U$. Therefore we need something looser than the strong ambiguity requirement: $\mu \equiv 2^\mu$.

To be more explicit, let us recall that it follows from [7] that if $\mu = |M|$ is ambiguous, there is a model $\mathcal{N} = \langle N, \in_\mathcal{N} \rangle$ of NF —whence of NFU— such that the typed structure $\langle N, \mathcal{P}^\mathcal{N}(N), \mathcal{P}^\mathcal{N}(\mathcal{P}^\mathcal{N}(N)), ...; \in_\mathcal{N} \rangle = \langle N, N, N, ...; \in_\mathcal{N} \rangle$ is elementarily equivalent to $\langle\langle M \rangle\rangle$. If we are searching for models of NFU, the ambiguity requirement can be relaxed, however. In fact, as is shown in [4], if $2^\mu \leq \kappa \equiv \mu$ and $\mu = |M|$, there is a model $\mathcal{N} = \langle N, \in_\mathcal{N} \rangle$ of NFU such that $\langle N, \mathcal{P}^\mathcal{N}(N), \mathcal{P}^\mathcal{N}(\mathcal{P}^\mathcal{N}(N)), ...; \in_\mathcal{N} \rangle$ and $\langle\langle M \rangle\rangle$ are elementarily equivalent. So that the natural way to extract a right notion of ambiguous cardinal out of NFU is to consider ambiguous pairs: $\langle \mu, \kappa \rangle$ is ambiguous iff $2^\mu \leq \kappa$ and $\mu \equiv \kappa$. It will then further be possible to replace the "jump" 2^μ by a generalized one.

Definitions

A typed function *F is a functional (class function) defined on cardinals such that $F(|x|)$ is a definable cardinal in $\langle\langle x \rangle\rangle$ in the following sense: there is a typed formula $A(y^n)$, with y^n as sole free variable, such that $\{ v(y^n) \in \mathcal{P}^n(x) \mid \langle\langle x \rangle\rangle, v \models A(y^n) \}$*[8] *is a cardinal in the sense of type theory and $F(|x|)$ is the corresponding cardinal in the sense of* ZF*. In other words, $\{ v(y^n) \in \mathcal{P}^n(x) \mid \langle\langle x \rangle\rangle, v \models A(y^n) \}$ is an equivalence class of the equipollence relation, restricted to the members of $\mathcal{P}^n(x)$, and $F(|x|)$ is the cardinal of each of its elements*[9]*.*

A jump *J is a typed increasing progressive function such that the Hartogs' number of its argument can be exceeded by a given, finite iteration:*

- *if $\alpha \leq \beta$ then $J(\alpha) \leq J(\beta)$;*
- *$\alpha < J(\alpha)$;*
- *there is a (concrete) natural number q such that, for all α, $\aleph(\alpha) \leq J^q(\alpha)$.*

Examples of jumps are: 2^x, i.e. $\{ v(y^2) \in \mathcal{P}(\mathcal{P}(x)) \mid \langle\langle x \rangle\rangle, v \models |y^2| = \nu^+ \}$; 2^{2^x}; $x + \aleph(x)$; $\aleph(x)$ *if x is well-ordered and* 2^x *otherwise, ...*

$\aleph(x)$ *is a typed function, but not necessarily a jump, since* ZF $\not\vdash \mu < \aleph(\mu)$.

[8] $\langle\langle x \rangle\rangle, v \models A$ means that valuation v satisfies A in the model $\langle\langle x \rangle\rangle$.

[9] Naively, a typed function is a function whose value can be calculated by people living in a finite segment of the typed structure over a set of the size of the argument.

If J is a jump, then:
$\Phi^J(\alpha) = \{\alpha, J(\alpha), J^2(\alpha), ...\}$;
and $\Phi^J_\mu(\alpha) = \Phi^J(\alpha) \cap \{\, \beta \mid \beta \leq \mu \,\}$.

$\langle \mu, \kappa \rangle$ *is* J-ambiguous *iff* $J(\mu) \leq \kappa$ *and* μ *is indiscernible from* κ*. Thus, "*μ *is ambiguous" means that* $\langle \mu, 2^\mu \rangle$ *is* $2^{\cdots}$*-ambiguous.*

Theorem 3 *If* $\langle \mu, \kappa \rangle$ *is* J*-ambiguous, then there exists* $\delta \in \mathsf{WG}^J(\kappa)$ *such that* $\delta \not\leq \mu$ *and* $\Phi^J_\kappa(\delta)$ *is infinite.*

Theorem 4 *If* $\langle \mu, \kappa \rangle$ *is* J*-ambiguous and if* $J(\aleph(\mu))$ *is well-ordered, then* $\Phi^J_\kappa(\aleph(\mu))$ *is infinite.*

We conclude by quoting the ZF-content of Boffa's last result (from [1]):

> If $\langle \mu, \kappa \rangle$ is J-ambiguous and $\Phi^J_\kappa(\mu)$ is finite, then κ is not well-ordered.

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